Compactified Spherically Symmetric Black-hole Spacetimes in Analytical Coordinates

J. Haláček and T. Ledvinka
Charles University, Faculty of Mathematics and Physics, Prague, Czech Republic.

Abstract. In this paper we show available Penrose-Carter compactifications of the Schwarzschild space-time, their properties and analytical structure, namely the problems they exhibit at a space-time infinity. We have created a method of compactification of coordinates based on the analytical requirements which a compactified spherically symmetric metrics must satisfy. We use this procedure to construct the analytical compactification of the Schwarzschild and Reissner-Nordström space-times. Even though the metric functions are written using a Lambert function and other standard transcendental functions, to prove the analyticity of the compactified metrics we use a theorem on differential equations in complex variable.

Introduction

A very useful tool used to discuss various aspects of black-hole spacetimes are the so-called Penrose-Carter diagrams. They are used to illustrate a structure of horizons, worldlines of various observers and null particles or e.g. global properties of spacetime slicings. In textbooks [Misner, Thorne, and Wheeler, 1973; Wald, 1984; Novikov and Frolov, 1998] the standard depiction of the compactified Schwarzschild spacetime is usually based on the Kruskal construction of the maximal extension of the Schwarzschild metric (Figure 1). Since many processes are discussed from the point of view of very distant observers or in terms of null radiation which escapes to infinity, some insight may be lost as this version of compactification of Schwarzschild spacetime does not respect the fact that at these infinities the spacetime should resemble the Minkowski one. Its Penrose-Carter diagram is known to look like a patch on a manifold of Einstein universe inflated using an appropriate conformal factor into infinite empty space.

In this contribution we construct in a similar manner Lorenzian manifold which conformally scaled Schwarzschild black-hole spacetime is a part of. In the first two sections we introduce necessary notions and standard approaches. We also point out the shortcomings of existing compactification transformations when describing the Schwarzschild metric as an asymptotically flat spacetime. In the third section we show how to construct coordinates analytically covering complete Schwarzschild spacetime beyond future/past null infinities $\mathcal{I}^\pm$ and discuss their properties.

Conformal compactification of asymptotically flat spacetimes

The principles of a compactification can be easily demonstrated on the Minkowski spacetime where using coordinates $u_M$ and $v_M$ given by the transformation

\[ 2r_M = \tan v_M - \tan u_M, \quad 2t_M = \tan v_M + \tan u_M. \]

(1)

Then the usual Minkowski line element in spherical coordinates $ds^2_M = -dt_M^2 + dr_M^2 + r_M^2 d\omega^2$ (where $d\omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$) transforms into

\[ ds^2_M = \frac{1}{\cos^2 u_M \cos^2 v_M} \left( -dU_M dV_M + \frac{1}{4} \sin^2 (v_M - u_M) d\omega^2 \right), \]

(2)

where $u_M, v_M \in [-\pi/2, \pi/2]$, which implies that the infinity of Minkowski spacetime appears at a finite values of coordinates $u_M, v_M$ and according to endpoints of spacelike, null and timelike geodesics, this infinity can be decomposed as $\mathcal{I} = i^0 \cup \mathcal{I}^\pm \cup i^\pm$ (where $i^0$ resp. $i^\pm$ means spacelike resp. future/past timelike infinity).

The standard choice $\Omega_M = \cos u_M \cos v_M$ can be clearly identified in (2) and then using the same coordinates $\tilde{u}_M = u_M, \tilde{v}_M = v_M$ we obtain a larger, non-physical manifold $(\tilde{M}, \tilde{ds}^2)$ (of the Einstein static universe [Wald, 1984])

\[^1\text{Tilde denote quantities associated with conformal metric $\tilde{ds}^2 = \Omega^2 ds^2$.}\]
Standard compactification of the Schwarzschild spacetime

The Schwarzschild metric written in the Schwarzschild coordinates \( t, r, \theta, \phi \) has the line element

\[
\mathrm{d}s^2 = - \left(1 - \frac{2M}{r}\right) \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}} + r^2 \sin^2 \theta \, \mathrm{d}\theta^2 + r^2 \, \mathrm{d}\phi^2
\]

with \( \mathrm{d} \omega^2 = \mathrm{d}\theta^2 + \sin^2 \theta \, \mathrm{d}\phi^2 \). Since both the Kruskal’s construction of horizon penetrating coordinates \([Kruskal, 1960]\) and the notion of null infinity are based on the null radial geodesics, we introduce the tortoise coordinate

\[
r^* = f(r) = r + 2M \ln \left(\frac{r}{2M} - 1\right).
\]

As we will show later its analytic properties in detail, it is explicitly denoted as a function \( f \) of Schwarzschild radius \( r \). Using the tortoise coordinate the radial null geodesics are naturally described by two parameters \( u, v \) such that the function \( x^r(u, v) \) given by

\[
\frac{r^*}{2} = \frac{1}{2} (v - u), \quad t = \frac{1}{2} (v + u)
\]

and spherical coordinates \( \theta, \phi \) represents (i) an outgoing radial null geodesic parametrized by \( u \) and labeled by \( v, \theta, \phi = \text{const.} \) and (ii) an ingoing radial null geodesic parametrized by \( v \) and labeled by \( u, \theta, \phi = \text{const.} \). The same formulae can be understood also as a coordinate transformation which brings the line element of the Schwarzschild spacetime into the form

\[
\mathrm{d}s^2 = - \left(1 - \frac{2M}{r(u, v)}\right) \mathrm{d}u \, \mathrm{d}v + r(u, v)^2 \mathrm{d}\omega^2.
\]
In textbooks, this is a starting point from which the Kruskal’s maximal extension of the Schwarzschild metric is derived. There infinites $u \to \infty$ and $v \to -\infty$ are studied and moved by an appropriate transform to finite values of Kruskal coordinates. In a very similar way, the process of compactification of an asymptotically flat spacetime is based on moving the null infinities $u \to -\infty$ and $v \to +\infty$ to the inner points of a larger unphysical manifold with coordinates $\tilde{u}, \tilde{v}$. We will show, that both infinites of $u,v$ coordinates can be treated at once obtaining this way the analytic conformal extension of the Kruskal’s maximal extension of the Schwarzschild spacetime.

The usual two null Kruskal coordinates $u,v \in \mathbb{R}$ are restricted so that $r(u,v) > 0$, events very (infinitely) distant in time or space are described by (infinitely) large coordinates $u,v$. To fit the whole spacetime into an illustration, their compactification $u = -4M \ln (-\tan u_1)$, $v = 4M \ln (\tan v_1)$ introduced by Misner, Thorne, and Wheeler [1973] is usually used. Using these coordinates the Schwarzschild line element reads

$$ds^2_1 = -\frac{1}{\cos^2 u_1 \cos^2 v_1} \frac{32M^3}{r(u_1,v_1)} e^{-\frac{2(u_1^1+v_1^1)}{M}} du_1 dv_1 + r(u_1,v_1)^2 d\omega^2,$$

where implicit dependent function $r(u_1,v_1)$ is defined in (12).

The coordinates $u_1,v_1$ are related to the Schwarzschild coordinates $t,r$ by the transformation

$$f(r(u_1,v_1)) = 2M \ln (\tan u_1 \tan v_1),$$

$$t = \pm 2M \ln \left| \frac{\tan u_1}{\tan v_1} \right|$$

where for $r < 2M$ both sides of the first equation contain a term $2M\pi i$ so it becomes a complex-valued implicit function prescription for a real function of two real variables $r(u_1,v_1)$. The fact that the Kruskal (here compactified) maximal extension can be found is an implication of the existence of a function $r(u_1,v_1)$ with desired analytic properties since metric coefficients in (11) are then simple analytic functions of $u_1,v_1$ and $r(u_1,v_1)$.

The easiest way to show that $r(u_1,v_1)$ is analytic on horizons $\mathcal{H}^\pm$ can be based on the fact that the Lambert function $W_0(z)$ using which we can write

$$r(u_1,v_1) = 2M \left( 1 + W_0(-e^{-1} \tan u_1 \tan v_1) \right)$$

is analytic on $z \in (-e^{-1}, \infty)$ [Weissstein, 2011]. Thus the line element (11) is analytic at the domain where $-\infty < \tan u_1 \tan v_1 < 1$. The way the Schwarzschild coordinates $r,t$ cover this domain can be seen in Figure 1.

It is known that the exponential term $e^{-r(u_1,v_1)/2M}$ in the Kruskal line element (11) prohibits one to satisfy conditions (5-6). Thus several coordinate transformations were given in literature [Novikov and Frolov, 1998; Hervik and Grøn, 2007] which should lead to $(M,ds^2)$ with the behavior more similar to that of Minkowski spacetime near $\mathcal{I}$. Namely, as the adjective conformal should mean angle-preserving, they preserve the way “lines” $r \to \text{const.}$ and $t \to \text{const.}$ behave near $\mathcal{I}^\pm$.

As an example of hurdles this approach presents, let us use the transformation proposed by Novikov and Frolov [1998]:

![Figure 1. The compactified Kruskal diagram of the Schwarzschild spacetime.](image-url)
where \( \alpha \) and \( \beta \) are yet unknown parameters for which both \( \alpha,\beta > 0 \) are positive. The Schwarzschild line element then takes form (5-6).

If the conformal factor \( \Omega \) and we see that analytic coverage of horizon where \( \alpha,\beta > 0 \) is desired, the compactification of Schwarzschild line element then takes form (19).

\[
\sqrt{\alpha} = \frac{1}{\Omega} \frac{\sqrt{\alpha} \cos \omega}{\sqrt{\omega} \omega} (\Omega U,V) = \Omega U,V),
\]

where \( \alpha,\beta \) are rational functions in \( \sin x, \cos x \) and we see that \( \beta \) should become negative where \( r - 2M < 0 \). From (22) we can write

\[
r(U,V) = 2M \left[ 1 + W_0 \left( \beta(V)\beta(-U)^{\frac{\alpha Y + \beta(-U)}{2M}} - 1 \right) \right]
\]

and see that analytic covering of horizon where \( U,V \approx 0 \) requires \( \beta(x) \approx x \).

We have to further restrict \( \alpha,\beta \) so that \( \sqrt{\alpha} \) is analytic near \( \mathcal{J}^\pm \). While the properties of Kruskal coordinates show, that \( \alpha \) may be omitted if only analytic coverage of horizons is desired, the compactification and analytic imbedding of regions near \( \mathcal{J}^\pm \) achieved by Ashtekar, Hansen [1978] shows we might need \( \alpha \approx r \) there. Since \( \tilde{g}_{ab} \approx r(U,V)^2 \Omega_u^2 \) we decompose \( \Omega_u \approx U^{-1} \Omega_3(\alpha(U) + \alpha(-U)) + \Omega_3(r(U,V) - \alpha(U) - \alpha(-U)) \) and require that both terms are analytic. Thus we need \( \alpha(U) \approx 1/\Omega_3 \) near \( \mathcal{J}_+ \) and \( \alpha(-U) \approx 1/\Omega_3 \) near \( \mathcal{J}^- \). Then we define an auxiliary function \( \psi(U,V) \) describing the difference

\[
r(U,V) - \alpha(V) - \alpha(-U) = 2M \ln \frac{\beta(V)\beta(-U)}{r(U,V)} = 2M \left( 1 + \psi(U,V) \right)
\]

which must be an analytic function of both \( U \) and \( V \) at \( \mathcal{J}^\pm \) to make \( \Omega_3(r(U,V) - \alpha(V) - \alpha(-U)) \) \( \in \mathcal{C}^\omega \) there. This definition can be written as an implicit equation for \( \psi(U,V) \)

\[
\psi(U,V) = -1 + \ln \left( \frac{2M\beta(V)\beta(-U)}{\alpha(U)\alpha(-U)} \right) \frac{\alpha(V)\alpha(-U)}{1 + \frac{2M\beta(V)\beta(-U)}{\alpha(U)\alpha(-U)} \psi(U,V)}.
\]
which determines the ratio $\beta(V)/\alpha(V) \sim 1$ near $\mathcal{I}^+$. We found that a reasonably simple choice of functions $\alpha$ and $\beta$ leading to

$$g(x) = \frac{M}{\cos x} + 2M \ln \frac{1 - \cos x}{\sin x \cos x}$$

(26)

together with the conformal factor

$$\Omega_3(U, V) = \cos U \cos V \sqrt{4M^2 \sqrt{(1 + \cos U)^2 - 2\cos^3 U} \sqrt{(1 + \cos V)^2 - 2\cos^3 V}}$$

(27)

satisfies all the necessary conditions. So the resulting metric has form

$$\tilde{d}s^2_3 = \left(1 - \frac{2M}{r(U, V)}\right) \sin U \sin V dU dV + \Omega_3^2 r(U, V)^2 d\omega^2.$$

(28)

Using (23) and properties of Lambert function $W_0(x)$ at $x = 0$ it is easy to show, that the metric (28) is analytic on $\mathcal{H}^\pm$. Since to use the same method on $\mathcal{I}^\pm$ we would need to regularize the expression of type $W_k(\infty e \infty) - \infty$, we rather write an implicit equation

$$\psi(U, V) = \ln \frac{\sin U \sin V (\cos U + \cos V + 2\cos U \cos V \psi(U, V))}{2(1 - \cos V)(\cos U - 1)}.$$

(29)

We can demonstrate that $\psi(U, V)$ is analytic e.g. at $\mathcal{I}^+$ is to put $\psi_t(U) \equiv \psi(U, V) - (2\cos U)^{-1}$ and differentiate (29) with respect to $V$. This way we get a differential equation

$$\frac{d\psi_t(U)}{dV} = \sqrt{\frac{1 - \cos V}{\sin V}} - \sqrt{\frac{\cos V - 2\cos^2 V - 2\psi_t(U)}{2(\psi_t(U) + 1) \cos V + 1}}.$$

(30)

Because its right-hand side is analytic in $V$ and $\psi_t(U)$ it has an analytic solution near $\mathcal{I}^+$ for initial condition implied by limit of (29) there. We have used a theorem about existence of the solution of differential equation [e.g., Dieudonné, 1969, section 4].

**Application of the method on the Reissner–Nordström spacetime**

We can use the latter method for another spherically symetric spacetime, namely the Reissner–Nordström spacetime. More details about the spacetime can be found in [Misner, Thorne, and Wheeler, 1973].

Because of presence of two horizons we have to slightly modify our transformation function. We have to "comensate" another logarithmic part at a Regge–Wheeler coordinate in the Reissner–Nordström
The Carter–Penrose diagram of the Reissner–Nordström spacetime covering its analytical extension beyond $\mathcal{I}^\pm$. We know the spacetime have to be covered by two sets of a coordinate maps so we consider only the outer part of the spacetime i.e. $r > r^-$. This modification leads to the function

$$g_{RN}(x) = \frac{M}{\cos x} + \frac{r^+}{r^+ - r^-} \ln \left( \frac{1 - \cos x}{\cos x \sin x} \right) - \frac{r^-}{r^+ - r^-} \ln \left( \frac{1}{\cos x} \right),$$

(31)

where $r^\pm = M \pm \sqrt{M^2 - Q^2}$ and the conformal metric reads

$$\tilde{d}s^2 = \frac{1 - \frac{2M}{\tau(U, V)} + \frac{Q^2}{\tau(U, V)^2}}{4M^2 \sin U \sin V} dU dV + \Omega_{RN}^2 r(U, V)^2 d\omega^2.$$

The point of this section is to illustrate our method for another stationary and asymptotically flat spacetime so we do not write the conformal factor explicitly, but it is easy to calculate it. For proving of analyticity one can use the same method like in the previous section. Penrose–Carter diagram for Reissner–Nordström spacetime can be seen in Figure 3.

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References


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We are using the same letters $U, V$ for coordinates as in previous section, but now we consider different spacetime, so there is no possibility of misunderstanding.