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The van Hiele Model of Geometric Thinking

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Abstract. The van Hiele theory describes how young people learn geometry. It postulates five levels of geometric thinking which are labeled visualization, analysis, abstraction, formal deduction and rigor. Each level uses its own language and symbols. Students or pupils pass through the levels "step by step". This hierarchical order helps them to achieve better understanding and results. This article presents an overview of the model. It is focused on possibilities how to apply this theory on Czech mathematical education.

Introduction

Pierre van Hiele and his wife Dina van Hiele-Geldof were Dutch researchers and teachers. They had personal experience with difficulties which their students had in learning geometry. Therefore, they dealt with these problems in detail. The theory originated in their theses at the University of Utrecht in 1957. Pierre van Hiele devoted his lifetime to their theory, Dina died shortly after completing her thesis.

Research based on the theory was carried out in the Soviet Union in the 1960s. Using its results, a very successful new geometry curriculum was designed in the Soviet Union. American researchers did several large studies on the van Hiele theory in the late 1970s [*Usiskin, 1982 and Senk, 1985*]. These studies influenced American NCTM Standards and Common Core State Standards.

Van Hiele theory

The theory has three aspects: the existence of levels, the properties of the levels, and the progress from one level to the next level.

Van Hiele levels

According to the theory, there are five levels of thinking or understanding in geometry:

- Level 0 Visualization
- Level 1 Analysis
- Level 2 Abstraction
- Level 3 Deduction
- Level 4 Rigor

Originally van Hieles numbered them from 0 to 4, the USA introduced numbering from 1 to 5; later Pierre van Hiele used only 3 levels. Moreover, also the level labels vary at present.

Level 0 Visualization (Basic visualization or Recognition)
At this level pupils use visual perception and nonverbal thinking. They recognize geometric figures by their shape as "a whole" and compare the figures with their prototypes or everyday things ("it looks like door"), categorize them ("it is / it is not a…"). They use simple language. They do not identify the properties of geometric figures.

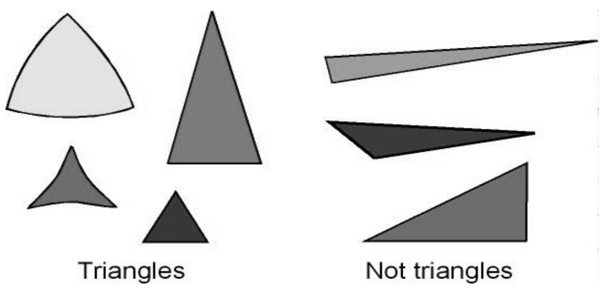

Figure 1. Children at Level 0 categorize triangles.

Level 1 Analysis (Description)
At this level pupils (students) start analyzing and naming properties of geometric figures. They do not see relationships between properties, they think all properties are important (= there is no difference between necessary and sufficient properties). They do not see a need for proof of facts discovered empirically. They can measure, fold and cut paper, use geometric software etc.

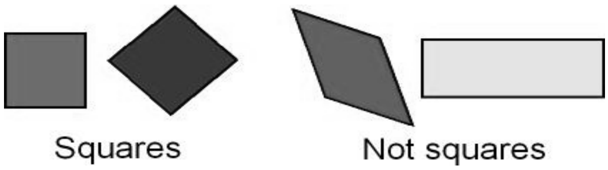

Figure 2. Children at Level 1 identify only one of the properties of squares.

Level 2 Abstraction (Informal deduction or Ordering or Relational)
At this level pupils or students perceive relationships between properties and figures. They create meaningful definitions. They are able to give simple arguments to justify their reasoning. They can draw logical maps and diagrams. They use sketches, grid paper, geometric SW.

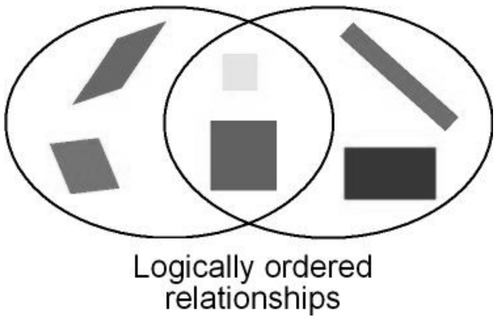

Figure 3. Children at Level 2 can draw a logical map of parallelograms.

Pierre van Hiele wrote: "My experience as a teacher of geometry convinces me that all too often, students have not yet achieved this level of informal deduction. Consequently, they are not successful in their study of the kind of geometry that Euclid created, which involves formal deduction."

Level 3 Deduction (Formal deduction)
At this level students can give deductive geometric proofs. They are able to differentiate between necessary and sufficient conditions. They identify which properties are implied by others. They understand the role of definitions, theorems, axioms and proofs.

Level 4 Rigor
At this level students understand the way how mathematical systems are established. They are able to use all types of proofs. They comprehend Euclidean and non-Euclidean geometry. They are able to describe the effect of adding or removing an axiom on a given geometric system.

Properties of levels

The levels have five important characteristics:

Fixed sequence (order)
A student cannot be at level N without having gone through level (N−1). Therefore, the student must go through the levels in order.

Adjacency
At each level, what was intrinsic in the preceding level becomes extrinsic in the current level.

Distinction
Each level has its own linguistic symbols and its own network of relationships connecting those symbols. The meaning of a linguistic symbol is more than its explicit definition; it includes the experiences which the speaker associates with the given symbol. What may be "correct" at one level is not necessarily correct at another level.

Separation
Two persons at different levels cannot understand each other. The teacher speaks a different "language" to the student at a lower level. The van Hieles thought this property was one of the main reasons for failure in geometry.
Attainment
The learning process leading to complete understanding at the next level has five phases – information, guided orientation, explanation, free orientation, integration, which are approximately not strictly sequential.

Five phases of the learning process

Van Hieles believed that cognitive progress in geometry can be accelerated by instruction. The progress from one level to the next one is more dependent upon instruction than on age or maturity. They gave clear explanations of how the teacher should proceed to guide students from one level to the next. However, this process takes tens of hours.
Information or Inquiry
Students get the material and start discovering its structure. The teacher holds a conversation with the pupils, in well-known language symbols, in which the context he wants to use becomes clear.
(A teacher might say: "This is a rhombus. Construct some more rhombi on your paper.")
Guided or directed orientation
Students deal with tasks which help them to explore implicit relationships. The teacher suggests activities that enable students to recognize the properties of the new concept. The relations belonging to the context are discovered and discussed.
(A teacher might ask: "What happens when you cut out and fold the rhombus along a diagonal? Along the other diagonal?")
Explanation or Explication
Students formulate what they have discovered, and new terminology is introduced. They share their opinions on the relationships they have discovered in the activity. The teacher makes sure that the correct technical language is developed and used. The van Hieles thought it is more useful to learn terminology after students have had an opportunity to become familiar with the concept.
(A teacher might say: "Here are the properties we have noticed and some associated terminology for the things you have discovered. Let us discuss what these mean: The diagonals lie on the lines of symmetry. There are two lines of symmetry. The opposite angles are congruent. The diagonals bisect the vertex angles.")
Free orientation
Students solve more complex tasks independently. It brings them to master the network of relationships in the material. They know the properties being studied, but they need to develop understanding of relationships in various situations. This type of activity is much more open-ended.
(A teacher might say: "How could you construct a rhombus given only two of its sides?" and other problems for which students have not learned a fixed procedure.)
Integration
Students summarize what they have learned and keep it in mind. The teacher should give to the students an overview of everything they have learned. It is important that the teacher does not present any new material during this phase, but only a summary of what has already been learned.
(A teacher might say: "Here is a summary of what we have learned. Write this in your notebook and do these exercises for homework.")

Pierre van Hiele wrote: "A definition of a concept is only possible if one knows, to some extent, the thing that is to be defined."

American study based on van Hiele theory

The fundamental purpose of the American study [*Usiskin, 1982*] was to test the ability of the van Hiele theory to describe and predict the performance of students in secondary school geometry. The tested sample consisted of 2699 students enrolled in a one-year geometry course in 13 schools. There were represented students from the seventh through the twelfth grades, 96 % of students were between the ages of 14 to 17. Van Hiele Level Test was set in the first week of school year and again four weeks before the end of school. Apart from that were set other tests (Entering Geometry Test, Comprehensive Assessment Program Geometry Test and Proof Test). Van Hiele Level Test contained 25 questions, 5 for each level. It was a multiple-choice test.

Main results were:

- In the form given by the van Hieles, levels 0 – 3 are easily testable, but level 4 either does not exist or is not testable.
- Van Hiele level is a very good predictor of performance in the standard test and in the proof test.
- Almost half of the students were placed in a course in which their chances of being successful at proof were only 50 to 50.

Conclusion

Currently there are two lines of research based on van Hiele theory in the world. Many researchers want to transfer van Hiele theory to other areas of mathematics, for example Boolean Algebra, Function – Analysis – Calculus etc. Further studies are done in the field of using dynamic geometry SW to achieve higher van Hiele levels.

Although many studies all over the world demonstrated that van Hiele theory can help improve geometric understanding, it was not taken into account in Czech mathematical education. Traditional teaching methods often involve only the *Integration* phase, which explains why students do not master the material. Teachers believe they express themselves clearly and logically, but their reasoning is not understandable to students at lower levels. We cannot be sure whether most Czech secondary students are *at least at Level 2* before entering Geometry course. Czech textbooks and exams should be adapted to van Hiele theory, preferably after relevant research. This should be the aim of my own thesis.

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