The Zeroth Symplectic Twistor Operator

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Abstract. We focus our attention to one of the so called symplectic twistor operators acting on exterior differential forms with values in symplectic spinors. We introduce this operator and study it on a real two dimensional plane considered as a symplectic manifold and we present a part of its kernel.

Introduction

The research in the symplectic spinor geometry, which is an analogue of the Riemannian spinor geometry, was initiated by D. Shale [Shale, 1962] and B. Konstant [Kostant, 1974]. K. Habermann in [Habermann, 1995] continued in their work and introduced the symplectic Dirac operator. See the monograph [Habermann, 2006], where you can find an introduction to the symplectic spinor geometry and symplectic Dirac operators.

The basic ingredient in this growing area of non-compact geometry is the Segal-Shale-Weil representation, a faithful infinite dimensional unitarizable representation of the metaplectic group. Another ingredient is a symplectic manifold (M, ω). If there is double covering of the symplectic frame bundle of M, the so called metaplectic bundle, one can induce from the Segal-Shale-Weil representation to obtain the symplectic spinor bundle. This bundle is of infinite rank and its fibers are vector spaces isomorphic to the Segal-Shale-Weil representation.

To a symplectic connection ∇ on TM one associates the so called symplectic spinor covariant derivative ∇S, which acts on sections of the symplectic spinor bundle. The other operators derived from this connection can be obtained by standard procedures, see e.g., Kadlčáková [Kadlčáková, 2001] or Krýsl [Krýsl, 2006].

In the present paper we deal with the zeroth symplectic twistor operator T0, which maps symplectic spinor fields into symplectic spinor differential forms. In the beginning we introduce and recall basic definitions and notions. Then we introduce T0 and prove its invariance. At the end, we derive a formula for the zeroth symplectic twistor operator in dimension two and present a few of its solutions.

Basic terms and constructions

Let (V, Ω) be a 2n dimensional real symplectic vector space equipped with canonical symplectic form Ω = dp1 ∧ dq1 + ... + dpn ∧ dqn. Let L and L′ be two isotropic subspaces of V such that L ⊕ L′ = V. We choose a symplectic basis {∂/∂q1, ..., ∂/∂q2n, ∂/∂p1, ..., ∂/∂pn} adapted to the above splitting, i.e., the first n elements are in L and the other n elements are in L′. Let us denote {e1, ..., en, e_n+1, ..., e_2n} the chosen symlectic basis and the dual basis by {e1, ..., e_2n}. We will be using the symbol ω jl for inverse matrix to ω ji = Ω(e_j, e_i).

Let (M2n, ω) be a symplectic manifold and let πP : P → M be the symplectic frame bundle over the symplectic manifold (M, ω). The symplectic group Sp(2n, R) has a unique connected double covering group Mp(2n, R), which is called the metaplectic group. We denote this double covering by λ : Mp(2n, R) → Sp(2n, R). Let S be the Schwartz space of rapidly decreasing functions R^n → C equipped with its usual locally convex complete (Fréchet) topology and denote by m : Mp(2n, R) → Aut(S) the metaplectic representation of the metaplectic group on S. It is well known that S = S_+ ⊕ S_- not only as a vector space but also as Mp(2n, R)-module. Here S_+ and S_- are spaces of even and odd functions in S, respectively. See [Weil, 1964] for more information on the Segal-Shale-Weil representation.

Assume that πQ : Q → M is a principal Mp(2n, R)-bundle over the symplectic manifold
$M$. The metaplectic structure on $(M, \omega)$ is principal $Mp(2n, \mathbb{R})$-bundle $\pi_Q : Q \to M$ together with a surjective 2:1 bundle homomorphism $\Lambda : Q \to P$ over the identity on $M$, which is chosen in such a way, that the following diagram commutes.

\[
\begin{array}{ccc}
Q \times Mp(2n, \mathbb{R}) & \longrightarrow & Q \\
\Lambda \times \Lambda & \downarrow & \Lambda \\
P \times Sp(2n, \mathbb{R}) & \longrightarrow & P
\end{array}
\]

\[\pi_Q \circ (\Lambda \times \Lambda) = \Lambda \circ \pi_P\]

The symplectic spinor bundle is a vector bundle associated to the $Mp(2n, \mathbb{R})$-bundle $Q$ via the metaplectic representation $m : Mp(2n, \mathbb{R}) \to \text{Aut}(S)$. Let us denote the symplectic spinor bundle by $S = Q \times m S$. Sections $\phi \in \Gamma(M, S)$ are called symplectic spinor fields.

We are interested in symplectic spinor valued exterior forms with values in the vector space $\bigwedge^* V^* \otimes S$, where $V^*$ is the dual vector space to $V$. The representation of $Mp(2n, \mathbb{R})$ on symplectic spinor valued forms

$$\rho : Mp(2n, \mathbb{R}) \to \text{Aut} \left( \bigwedge^* V^* \otimes S \right)$$

is defined by

$$\rho(g)(\alpha \otimes s) = (\lambda(g)^*)^{\wedge r} \alpha \otimes m(g)s,$$

where $\wedge^r$ denotes the $r$-th exterior power, $r = 0, \ldots, 2n$, $g \in Mp(2n, \mathbb{R})$, $\alpha \in \bigwedge^r V^*$ and $s \in S$. We extend it by linearity to any element. Furthermore, we call

$$\Omega^r(M, S) = \Gamma \left( M, Q \times_\rho \left( \bigwedge^r V^* \otimes S \right) \right)$$

the space of exterior differential forms with values in symplectic spinors.

The following decomposition into irreducible $\mathfrak{g}$-modules holds. For all $i = 0, \ldots, 2n$,

$$\bigwedge^i V^* \otimes S_\pm = \bigoplus_{(j,k) \in I_n} E^{ij}_{\pm}.$$

Here $I_n := \{(i,j)|i = 0, \ldots, n, j = 0, \ldots, i\} \cup \{(i,j)|i = n+1, \ldots, 2n, j = 0, \ldots, 2n - i\}$ and $\mathfrak{g}$ is the Lie algebra of $Mp(2n, \mathbb{R})$. In addition, for any $(i,j)$, $(i,k) \in I_n$, $j \neq k$ we have $E^{ij}_{\pm} \neq E^{ik}_{\pm}$ (as $\mathfrak{g}$-modules) for all combinations of $\pm$ on both sides. See [Krijsl, 2012] for a proof and description of $\mathfrak{g}$-modules $E^{ij}_{\pm}$ in terms of the representation theory.

Thus for $i = 0, \ldots, 2n$, the decomposition of the tensor product $\bigwedge^i V^* \otimes (S_+ \oplus S_-) = \bigoplus_{j,k} (E^{ij}_{+} \oplus E^{ij}_{-})$ is multiplicity-free. It means that $\bigwedge^i V^* \otimes (S_+ \oplus S_-)$ splits into non-isomorphic irreducible submodules $E^{ij}_{\pm}$. Let us set $E^{ij} = E^{ij}_{+} \oplus E^{ij}_{-}$ and consider the associated vector bundle $E^{ij} = Q \times_\rho E^{ij}$ for $(i,j) \in I_n$. There exist uniquely defined invariant projections

$$p^{ij} : \Omega^i(M, S) \to \Gamma(M, E^{ij})$$

for $(i,j) \in I_n$,

$$\nabla \omega = 0$$

and call it the symplectic connection. There is a unique correspondence between symplectic connections on $(M, \omega)$ and the connection 1-forms $Z$ on the principal $Sp(2n, \mathbb{R})$-bundle over $(M, \omega)$. Let $Z$ be a lift of the 1-form $Z$ to the principal $Mp(2n, \mathbb{R})$-bundle $\pi_Q : Q \to M$. From now on, we suppose the formulas hold only locally. Thus, let us choose a symplectic
frame \((U, \{e_j\}_{j=1}^{2n})\) and let us denote its dual co-frame by \((U, \{e^i\}_{j=1}^{2n})\). We hope that this will not cause any confusion. The symplectic spinor covariant derivative \(\nabla^S\) on the symplectic spinor fields is defined to be the covariant derivative associated to \(\bar{Z}\). See [Habermann, 2006] for the construction. The symplectic spinor exterior covariant derivative \(d^S\) is induced by the symplectic spinor covariant derivative \(\nabla^S\), i.e., for elements \(\alpha \otimes s \in \Omega^r(M, S)\), where \(\alpha \in \Omega^r(M)\) and \(s \in \Gamma(S)\), we have

\[
d^S(\alpha \otimes s) = da \otimes s + (-1)^r \sum_{i=1}^{2n} \epsilon^i \wedge \alpha \otimes \nabla^S_{e_i} s
\]

and extended by linearity.

Let us define symplectic twistor operators acting on symplectic spinor valued exterior forms. The definition is taken from [Krýsl, 2010]. Its contact projective analogue was introduced in [Kadlčaková, 2001].

**Definition 1.** For \(i = 0, \ldots, n - 1\), the symplectic twistor operators

\[
T_i : \Gamma(M, \mathcal{E}^{ii}) \to \Gamma(M, \mathcal{E}^{i+1,i+1})
\]

are defined by \(T_i := p^{i+1,i+1}d^S_{|\Gamma(M, \mathcal{E}^{ii})}\).

For \(i = n, \ldots, 2n - 1\), the symplectic twistor operators

\[
T_i : \Gamma(M, \mathcal{E}^{ii}) \to \Gamma(M, \mathcal{E}^{i+1,i-1})
\]

are defined by \(T_i := d^S_{|\Gamma(M, \mathcal{E}^{ii})}\).

**Proposition 2.** The zeroth symplectic twistor operator \(T_0\) has in a local basis the form

\[
T_0(s) = \left(1 + \frac{1}{n} \sum_{k=1}^{2n} \epsilon^k \otimes \nabla^S_{e_k} s + \frac{i}{n} \sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_j \cdot e_l \cdot \nabla^S_{e_k} s, \right)
\]

where \(s \in \Gamma(M, S)\), the dot \(\cdot\) denotes the symplectic Clifford multiplication explained in [Habermann, 2006] and \(i\) is the imaginary unit. We suppose the reader is acquainted with these notations.

**Proof.** See [Dostálková, 2011] for a proof, which is a direct computation. \(\square\)

Starting from now, we will work on the standard symplectic vector space \((\mathbb{R}^{2n}, \Omega)\). Note that all vector bundles on \(\mathbb{R}^{2n}\) are topologically trivial. The metaplectic group acts on the space \(\Omega^r(\mathbb{R}^{2n}, S)\) which is isomorphic to \(C^\infty(\mathbb{R}^{2n}, \wedge^r V^* \otimes S)\) via \([\hat{\rho}(g)f](x) = \rho(g)[f(\lambda(g^{-1})(x))]\), \(g \in Mp(2n, \mathbb{R})\), \(f \in C^\infty(\mathbb{R}^{2n}, \wedge^r V^* \otimes S)\) and \(x \in \mathbb{R}^{2n}\).

**Proposition 3.** The zeroth symplectic twistor operator \(T_0\) is \(Mp(2n, \mathbb{R})\)-invariant.

**Proof.** It is sufficient to show that

\[
\{T_0(\hat{\rho}(g)f)\}(x) = [\hat{\rho}(g)(T_0f)](x).
\]

Let \((x^1, \ldots, x^{2n}) = x \in \mathbb{R}^{2n}\) and \(f \in C^\infty(\mathbb{R}^{2n}, S)\). Due to the Proposition 2, the zeroth symplectic twistor operator has the form

\[
(T_0f)(x) = \left(1 + \frac{1}{n} \sum_{k=1}^{2n} \epsilon^k \otimes \frac{\partial}{\partial x^k} f(x) + \frac{i}{n} \sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_j \cdot e_l \cdot \frac{\partial}{\partial x^k} f(x). \right)
\]

(The symplectic spinor covariant derivative \(\nabla^S_{e_k}\) on \(\mathbb{R}^{2n}\) is the same as partial derivation in the corresponding variable.)

Let \(g \in Mp(2n, \mathbb{R})\) and let us compute the left side \(L\) of equation (2)
\[ L = \{ T_0[\bar{\theta}(g)f]\}(x) = T_0[\theta(g)f(\lambda(g)^{-1}x)] = T_0\{ m(g)[f(\lambda(g)^{-1}x)] \} \]

\[
= \left(1 + \frac{1}{n}\right) \sum_{k=1}^{2n} e^k \otimes m(g) \frac{\partial}{\partial x^k} [f(\lambda(g)^{-1}x)] \\
+ \frac{i}{n} \sum_{j,k,l=1}^{2n} e^l \otimes \omega^{kj} e_j \cdot e_l \cdot \left[ m(g) \frac{\partial}{\partial x^l} [f(\lambda(g)^{-1}x)] \right].
\]

Using the chain rule, we get

\[
L = \left(1 + \frac{1}{n}\right) \sum_{k,l=1}^{2n} e^k \otimes m(g) \left[ \left( \frac{\partial}{\partial x^f} \right) (\lambda(g)^{-1}x) [\lambda(g)^{-1}]^f_k \right] \\
+ \frac{i}{n} \sum_{j,k,l=1}^{2n} e^l \otimes \omega^{kj} e_j \cdot e_l \cdot \left[ m(g) \frac{\partial}{\partial x^l} (\lambda(g)^{-1}x) [\lambda(g)^{-1}]^h_k \right] \\
= \left(1 + \frac{1}{n}\right) \sum_{k,l=1}^{2n} e^k \otimes [\lambda(g)^{-1}]^l_k m(g) \left[ \left( \frac{\partial}{\partial x^f} \right) (\lambda(g)^{-1}x) \right] \\
+ \frac{i}{n} \sum_{j,k,l=1}^{2n} e^l \otimes [\lambda(g)^{-1}]^h_k \omega^{kj} e_j \cdot e_l \cdot m(g) \left[ \frac{\partial}{\partial x^l} f(\lambda(g)^{-1}x) \right].
\]

Now, let us expand the right hand side \( P \) of equation (2).

\[
P = [\bar{\theta}(g)(T_0 f)](x) = \theta(g)[(T_0 f)(\lambda(g)^{-1}x)] \\
= \theta(g) \left[ \left(1 + \frac{1}{n}\right) \sum_{k=1}^{2n} e^k \otimes m(g) \left[ \left( \frac{\partial}{\partial x^f} \right) (\lambda(g)^{-1}x) \right] \\
+ \frac{i}{n} \sum_{j,k,l=1}^{2n} e^l \otimes \omega^{kj} e_j \cdot e_l \cdot \left[ \frac{\partial}{\partial x^l} f(\lambda(g)^{-1}x) \right] \right] \\
= \left(1 + \frac{1}{n}\right) \sum_{k,m=1}^{2n} \lambda(g)^{-1} k^m m \otimes m(g) \left[ \left( \frac{\partial}{\partial x^f} \right) (\lambda(g)^{-1}x) \right] \\
+ \frac{i}{n} \sum_{j,k,l,o,p,q=1}^{2n} \omega^{kj} e_j \cdot e_l \cdot \left[ \lambda(g)^{-1} k^p m \otimes m(g) \left[ \left( \frac{\partial}{\partial x^l} \right) (\lambda(g)^{-1}x) \right] \right] \\
= \left(1 + \frac{1}{n}\right) \sum_{k,m=1}^{2n} \lambda(g)^{-1} k^m m \otimes m(g) \left[ \left( \frac{\partial}{\partial x^f} \right) (\lambda(g)^{-1}x) \right] \\
+ \frac{i}{n} \sum_{j,k,l,o,p,q=1}^{2n} \lambda(g)^{-1} k^p \omega^{kj} e_j \cdot e_l \cdot \left[ \lambda(g)^{-1} k^q m \otimes m(g) \left[ \left( \frac{\partial}{\partial x^l} \right) (\lambda(g)^{-1}x) \right] \right]
\]

For all \( A \in Sp(2n, \mathbb{R}) \) we have \( \Omega A^T = A^{-1} \Omega \), thus for \( \lambda(g) \in Sp(2n, \mathbb{R}) \) we obtain

\[
\sum_{j=1}^{2n} \omega^{kj} \lambda(g)^{T_j} j = \sum_{j=1}^{2n} \omega^{kj} \lambda(g)^{T_j} j = \sum_{r=1}^{2n} \lambda(g)^{-1} k^r \omega^{p_r}.
\]
which together with the previous computation implies

\[
P = \left( 1 + \frac{1}{n} \sum_{k,m=1}^{2n} \left[ (\lambda(g)^{-1})^{k} \mu^{m} \otimes \mu(g) \left[ \left( \frac{\partial}{\partial x^{k}} \right) (\lambda(g)^{-1} x) \right] \right] + \frac{i}{n} \sum_{k,p,q,r=1}^{2n} \epsilon^{p} \otimes \left[ (\lambda(g)^{-1})^{k} \mu^{p} \cdot \epsilon^{q} \cdot \mu(g) \left[ \left( \frac{\partial}{\partial x^{k}} \right) (\lambda(g)^{-1} x) \right] \right].
\]

Thus \( P = L \).

**Corollary 4.** The kernel of the zeroth symplectic twistor operator \( T_{0} \) on \( \mathbb{R}^{2n} \) is an \( Mp(2n, \mathbb{R}) \)-submodule of \( C^{\infty}(\mathbb{R}^{2n}, S) \).

**Proof.** Because of the form of the zeroth symplectic twistor operator and the definition of the topology on \( C^{\infty}(\mathbb{R}^{2n}, S) \) and \( C^{\infty}(\mathbb{R}^{2n}, (\mathbb{R}^{2n})^{*} \otimes S) \), one can prove that it is a continuous mapping from the Fréchet space \( C^{\infty}(\mathbb{R}^{2n}, S) \) into the Fréchet space \( C^{\infty}(\mathbb{R}^{2n}, (\mathbb{R}^{2n})^{*} \otimes S) \). Therefore \( Ker T_{0} \) is closed. Let us take \( a \in Ker T_{0} \). The zeroth symplectic twistor operator \( T_{0} \) is invariant under the representation \( \tilde{\rho} \) due to the previous Proposition. Thus, \( 0 = \tilde{\rho}(g)[T_{0}(a)] = T_{0}(\tilde{\rho}(g)(a)) \) is satisfied for every element \( g \in Mp(2n, \mathbb{R}) \). Therefore \( \rho(g)(a) \in Ker T_{0} \) and thus, \( Ker T_{0} \) is closed and also invariant under the action of the group \( Mp(2n, \mathbb{R}) \), i.e., it is a representation of \( Mp(2n, \mathbb{R}) \).

**Real two-dimensional case.**

Let us consider the two-dimensional flat case. Take \( (\mathbb{R}^{2}, \Omega) \) with coordinate \( x, y \), \( \Omega = dx \wedge dy \). Set \( \{e_{1}, e_{2}\} \) to be the symplectic basis of the tangent space to \( \mathbb{R}^{2} \). The basis elements act on symplectic spinors via \( e_{1} \cdot s = iqs \) and \( e_{2} \cdot s = \frac{\partial s}{\partial q} \). The symplectic spinor covariant derivative \( \nabla^{S}_{e_{k}} \) on \( \mathbb{R}^{2} \) is reduced to standard partial derivatives \( \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \).

We will understand symplectic spinors \( s \in \Gamma(\mathbb{R}^{2}, S) \) as the elements of \( C^{\infty}(\mathbb{R}^{3}, \mathbb{C}) \). In particular a symplectic spinor is a complex valued function of three real variables \( s(x,y,q) \in C^{\infty}(\mathbb{R}^{3}, \mathbb{C}) \) such that for fixed \( x, y \in \mathbb{R} \), the function \( s(x,y,\cdot) \) is an element of the Schwartz space \( S \).

**Proposition 5.** The zeroth symplectic twistor operator \( T_{0} : \Gamma(\mathbb{R}^{2}, S) \rightarrow \Gamma(\mathbb{R}^{2}, (\mathbb{R}^{2})^{*} \otimes S) \) has for \( s(x,y,q) \in C^{\infty}(\mathbb{R}^{3}, \mathbb{C}) \) the form

\[
T_{0}(s) = e_{1} \otimes \left( \frac{\partial s}{\partial x} - q \frac{\partial^{2} s}{\partial q \partial x} + iq \frac{\partial s}{\partial y} \right) + e_{2} \otimes \left( 2 \frac{\partial s}{\partial y} + iq \frac{\partial^{3} s}{\partial q^{2} \partial x} + q \frac{\partial^{2} s}{\partial q \partial y} \right).
\]

**Proof.** See [Dostálková, 2011] for a proof.

**Proposition 6.** The kernel of the zeroth symplectic twistor operator \( T_{0} \) is described by the equation

\[
\frac{\partial s}{\partial x} - q \frac{\partial^{2} s}{\partial q \partial x} + iq \frac{\partial s}{\partial y} = 0
\]

for functions \( s(\cdot, \cdot, \cdot) \in C^{\infty}(\mathbb{R}^{3}, \mathbb{C}) \) such that \( s(x,y,\cdot) \in S \), for each \( (x,y) \in \mathbb{R}^{2} \).

**Proof.** The statement is a consequence of the Proposition 5. Covectors \( e_{1} \) and \( e_{2} \) are independent. Any solution of the equation \( \frac{\partial s}{\partial x} - q \frac{\partial^{2} s}{\partial q \partial x} + iq \frac{\partial s}{\partial y} = 0 \) is solution of the equation \( 2 \frac{\partial s}{\partial y} + iq \frac{\partial^{3} s}{\partial q^{2} \partial x} + q \frac{\partial^{2} s}{\partial q \partial y} = 0 \).
It is not so hard to check that the functions

\[ s(x, y, q) = e^{-q^{2n+2}} q(x + 2nq^{2n}y), \quad (5) \]
\[ s(x, y, q) = e^{-\frac{q^2}{2}} (q^{n+1}(x + iy) - q^{n-1}(n)iy), \quad (6) \]
\[ s(x, y, q) = e^{-\frac{q^2}{2}} q(x + iy)^n, \quad (7) \]

where \( n \in \mathbb{N} \), satisfy the equation for the kernel of the zeroth symplectic twistor operator (Proposition 6). However, this is not a list of all functions in the kernel, since the kernel is a representation of the metaplectic group \( Mp(2, \mathbb{R}) \). See Corollary 4.

References


