Optimization Problems Under One-sided 
(max, min)-Linear Equality Constraints

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Abstract. In this article we will consider optimization problems, the objective function of which is equal to the maximum of a finite number of continuous functions of one variable. The set of feasible solutions is described by the system of (max, min)-linear equality constraints. First we will study the structure of the set of all solutions of the given system which can be solved by the appropriate formulation with equality constraints directly without replacing the equality constraints with the double inequalities (if |I| = m), which have to be taken into account. It arises an idea, whether it is not more effective to solve the problems with equality constraints directly without replacing the equality constraints with the double numbers of inequalities. In this article, we are going to propose such an approach to the equality constraints. First we will study the structure of the set of all solutions of the given system of equations with finite entries $a_{ij}$ & $b_i$, for all $i \in I$ & $j \in J$. Using some of the theorems characterizing the structures of the solution set of such systems, we will propose an algorithm, which finds an optimal solution of minimization problems with objective functions of the form: $f(x) \equiv \max_{j \in J} f_j(x_j)$, where $f_j$, $j \in J$ are continuous functions. Complexity of the proposed method of monotone or unimodal functions will be studied, possible generalizations and extensions of the results will be discussed.

1. Introduction

The algebraic structures in which (max, +) or (max, min) replace addition and multiplication of the classical linear algebra have appeared in the literature approximately since the sixties of the last century (see e.g. Butkovič and Hegedűs [1984], Cuninghame-Green [1979], Cuninghame-Green and Zimmermann [2001], and Vorobjov [1967]). A systematic theory of such algebraic structures was published probably for the first time in Cuninghame-Green [1979]. In recently appeared book Butkovič [2010] the readers can find latest results concerning theory and algorithms for the (max, +)-linear systems of equations. Gavalec and Zimmermann [2010] proposed a polynomial method for finding the maximum solution of the (max, min)-linear system. Zimmermann and Gad [2011] introduced a finite algorithm for finding an optimal solution of the optimization problems under (max, +)-linear constraints. Gavalec et al. [2012] provide survey of some recent results concerning the (max, min)-linear systems of equations and inequalities and optimization problems under the constraints described by such systems of equations and inequalities. Gad [2012] considered the existence of the solution of the optimization problems under two-sided (max, min)-linear inequalities constraints.

Let us consider the optimization problems under one-sided (max, min)-linear equality constraints of the form: \( \max_{j \in J} (a_{ij} \wedge x_j) = b_i, \quad i \in I, \) which can be solved by the appropriate formulation of equivalent one-sided inequality constraints and using the methods presented in Gavalec et al. [2012]. Namely, we can consider inequality systems of the form: \( \max_{j \in J} (a_{ij} \wedge x_j) \leq b_i, \quad i \in I, \) \( \max_{j \in J} (a_{ij} \wedge x_j) \geq b_i, \quad i \in I. \) Such systems have \( 2m \) inequalities (if \( |I| = m \)), which have to be taken into account. It arises an idea, whether it is not more effective to solve the problems with equality constraints directly without replacing the equality constraints with the double numbers of inequalities. In this article, we are going to propose such an approach to the equality constraints. First we will study the structure of the set of all solutions of the given system of equations with finite entries $a_{ij}$ & $b_i$, for all $i \in I$ & $j \in J$. Using some of the theorems characterizing the structures of the solution set of such systems, we will propose an algorithm, which finds an optimal solution of minimization problems with objective functions of the form: \( f(x) \equiv \max_{j \in J} f_j(x_j) \), where $f_j$, $j \in J$ are continuous functions. Complexity of the proposed method of monotone or unimodal functions $f_j$, $j \in J$ will be studied, possible generalizations and extensions of the results will be discussed.

2. One-sided (max, min)-Linear Systems of Equations

Let us introduce the following notations:

\[ J = \{1, \ldots, n\}, \quad I = \{1, \ldots, m\}, \] where $n$ and $m$ are integer numbers, \( R = (-\infty, \infty), \) \( \overline{R} = [-\infty, \infty], \) \( \overline{R}^n = \overline{R} \times \cdots \times \overline{R} \) (n-times), similarly \( \overline{R}^m = \overline{R} \times \cdots \times \overline{R}, \) \( x = (x_1, \ldots, x_n) \in \overline{R}^n, \) \( \alpha \wedge \beta = \min\{\alpha, \beta\}, \) \( \alpha \vee \beta = \max\{\alpha, \beta\} \) for any $\alpha, \beta \in \overline{R}$, we set per definition $-\infty \wedge \infty = -\infty,$
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\(-\infty \lor \infty = \infty, a_{ij} \in R, b_i \in R, \forall i \in I, j \in J\) are given finite numbers, and consider the following system of equations:

\[ \max_{j \in J}(a_{ij} \land x_j) = b_i, \quad i \in I, \]

\[ x \leq x \leq \bar{x}. \]

The set of all solutions of the system (1), will be denoted \(M^\infty\). Before investigating properties of the set \(M^\infty\), we will bring an example, which shows one possible application, which leads to solving this system.

**Example 2.1** Let us assume that \(m\) places \(i \in I \equiv \{1, 2, \ldots, m\}\) are connected with \(n\) places \(j \in J \equiv \{1, 2, \ldots, n\}\) by roads with given capacities. The capacity of the road connecting place \(i\) with place \(j\) is equal to \(a_{ij} \in R\). We have to extend for all \(i \in I, \ j \in J\) the road between \(i\) and \(j\) by a road connecting \(j\) with a terminal place \(T\) and choose an appropriate capacity \(x_j\) for this road. If a capacity \(x_j\) is chosen, then the capacity of the road from \(i\) to \(T\) via \(j\) is equal to \(a_{ij} \land x_j = \min(a_{ij}, x_j)\). We require that the maximum capacity of the roads connecting \(i\) to \(T\) via \(j \in J\) is equal to a given number \(b_i \in R\) and the chosen capacity \(x_j\) lies in a given finite interval i.e. \(x_j \in [\underline{x}_j, \bar{x}_j]\), where \(\underline{x}_j, \bar{x}_j \in R\) are given finite numbers. Therefore feasible vectors of capacities \(x = (x_1, x_2, \ldots, x_n)\) (i.e. the vectors, the components of which are capacities \(x_j\) having the required properties) must satisfy the system (1) and (2).

In what follows, we will investigate some properties of the set \(M^\infty\) described by the system (1). Also, to simplify the formulas in what follows we will set \(a_i(x) \equiv \max_{j \in J}(a_{ij} \land x_j) \forall i \in I\). Let us define for any fixed \(j \in J\) the set \(I_j = \{i \in I & a_{ij} \geq b_i\}, S_j(x_j) = \{k \in I & a_{kj} \land x_j = b_k\}\), and define the set \(M^\infty(x) = \{x \in M^\infty & x \leq \bar{x}\}\).

**Lemma 2.1** Let us set for all \(i \in I\) and \(j \in J\)

\[ T^\infty_{ij} = \{x_j : (a_{ij} \land x_j) = b_i \land x_j \leq \bar{x}_j\} \]

Then for any fixed \(i, j\) the following equalities hold:

(i) \(T^\infty_{ij} = \{b_i\}\) if \(a_{ij} > b_i \land b_i \leq \bar{x}_j\);

(ii) \(T^\infty_{ij} = [b_i, \bar{x}_j]\) if \(a_{ij} = b_i \land b_i \leq \bar{x}_j\);

(iii) \(T^\infty_{ij} = \emptyset\) if either \(a_{ij} < b_i\) or \(b_i > \bar{x}_j\).

**Proof:**

(i) If \(a_{ij} > b_i\), then \(a_{ij} \land x_j > b_i\) for any \(x_j > b_i\), also \(a_{ij} \land x_j < b_i\) for any \(x_j < b_i\), so that the only solution for equation \(a_{ij} \land x_j = b_i\) is \(x_j = b_i \leq \bar{x}_j\).

(ii) If \(a_{ij} = b_i \land b_i \leq \bar{x}_j\), then \(a_{ij} \land x_j < b_i\) for arbitrary \(x_j < b_i\), but \(a_{ij} \land x_j = b_i\) for arbitrary \(b_i \leq x_j \leq \bar{x}_j\).

(iii) If \(a_{ij} < b_i\), then either \(a_{ij} \land x_j = a_{ij} < b_i\) for arbitrary \(x_j \geq a_{ij}\), or \(a_{ij} \land x_j = x_j < b_i\) for arbitrary \(x_j < a_{ij}\). Therefore there is no solution for equation \(a_{ij} \land x_j = b_i\), which means \(T^\infty_{ij} = \emptyset\). Also if \(b_i > \bar{x}_j\), then either \(x_j \geq b_i > \bar{x}_j\), so that \(T^\infty_{ij} = \emptyset\), or \(x_j < b_i\) there are two cases, the first is \(x_j \leq a_{ij}\) and \(a_{ij} \land x_j = x_j < b_i\) and the second case is \(x_j > a_{ij}\) and \(a_{ij} \land x_j = a_{ij} < x_j < b_i\), therefore there is no solution for equation \(a_{ij} \land x_j = b_i\), which means \(T^\infty_{ij} = \emptyset\).
Lemma 2.2 Let us set for all $i \in I$ and $j \in J$

$$x_j^{(i)} = \begin{cases} b_i & \text{if } a_{ij} > b_i \& b_i \leq \pi_j, \\ \pi_j & \text{if } a_{ij} = b_i \& b_i \leq \pi_j \text{ or } T_{ij}^- = \emptyset, \end{cases}$$

and let

$$\hat{x}_j = \begin{cases} \min_{k \in I_j} x_j^{(k)} & \text{if } I_j \neq \emptyset, \\ \pi_j & \text{if } I_j = \emptyset, \end{cases}$$

Then we have

$$S_j(\hat{x}_j) = \{ k \in I : x_j^{(k)} = \hat{x}_j \}, \forall j \in J,$$

and the following statements hold:

(i) \( \hat{x} \in M^-(\bar{x}) \Leftrightarrow \bigcup_{j \in J} S_j(\hat{x}_j) = I \)

(ii) Let \( M^-(\bar{x}) \neq \emptyset \), then \( \hat{x} \in M^-(\bar{x}) \) and for any \( x \in M^-(\bar{x}) \Rightarrow x \leq \hat{x} \), i.e. \( \hat{x} \) is the maximum element of \( M^-(\bar{x}) \).

Proof:

(i) To prove the necessary condition we suppose \( \hat{x} \in M^-(\bar{x}) \), then \( \max_{j \in J}(a_{ij} \land \hat{x}_j) = b_i \), for all \( i \in I \). So that for all \( i \in I \), there exists at least one \( j(i) \in J \) such that \( a_{ij(i)} \land \hat{x}_{j(i)} = \max_{j \in J}(a_{ij} \land \hat{x}_j) = b_i \), then either \( \hat{x}_{j(i)} = b_i \), if \( a_{ij(i)} > b_i \& b_i \leq \pi_{j(i)} \), or \( \hat{x}_{j(i)} > b_i \), if \( a_{ij(i)} = b_i \& b_i \leq \pi_{j(i)} \). Hence for all \( i \in I \) there exists at least one \( j(i) \in J \) such that \( S_{j(i)}(\hat{x}_{j(i)}) \neq \emptyset \) and \( i \in S_{j(i)}(\hat{x}_{j(i)}) \Rightarrow \bigcup_{j \in J} S_j(\hat{x}_j) = I \).

To prove the sufficient condition, let \( \bigcup_{j \in J} S_j(\hat{x}_j) = I \), then for all \( i \in I \) there exists at least one \( j(i) \in J \) such that \( S_{j(i)}(\hat{x}_{j(i)}) \neq \emptyset \) and \( i \in S_{j(i)}(\hat{x}_{j(i)}) \). Therefore \( \hat{x}_{j(i)} = b_i \) if \( a_{ij(i)} > b_i \& b_i \leq \pi_{j(i)} \), then \( a_{ij(i)} \land \hat{x}_{j(i)} = \hat{x}_{j(i)} = b_i \). Otherwise \( \hat{x}_{j(i)} = \pi_{j(i)} \) if either \( a_{ij(i)} = b_i \& b_i \leq \pi_{j(i)} \), then \( a_{ij(i)} \land \hat{x}_{j(i)} = a_{ij(i)} = b_i \) or \( T_{ij(i)}^- = \emptyset \). Then for all \( i \in I \) there exists at least one \( j(i) \in J \) such that \( \max_{j \in J}(a_{ij} \land \hat{x}_j) = a_{ij(i)} \land \hat{x}_{j(i)} = b_i \). Then \( \hat{x} \in M^-(\bar{x}) \).

(ii) Let \( M^+(\bar{x}) \neq \emptyset \), then for each \( i \in I \), there exists at least one \( j(i) \in J \) such that \( T_{ij(i)}^+ \neq \emptyset \), and \( b_i \leq \pi_{j(i)} \& a_{ij(i)} \geq b_i \). Therefore there exists at least one \( j(i) \in J \) such that either \( a_{ij(i)} > b_i \& b_i \leq \pi_{j(i)} \), then \( x_{j(i)}^{(i)} = b_i \). Or \( a_{ij(i)} = b_i \& b_i \leq \pi_{j(i)} \), then \( x_{j(i)}^{(i)} = \pi_{j(i)} \so that \hat{x}_{j(i)} = b_i \) if \( i \in I_j \) and \( a_{ij(i)} \land \hat{x}_{j(i)} = b_i \) is satisfied. Otherwise if \( I_j = \emptyset \), we set \( \hat{x}_j = \pi_j \). Then \( \hat{x} \in M^+(\bar{x}) \) and for any \( x \in M^+(\bar{x}) \) we have \( x \leq \hat{x} \), i.e. \( \hat{x} \) is the maximum element of \( M^+(\bar{x}) \).

It is appropriate now to define \( M^+(\bar{x}, \bar{x}) = \{ x \in M^+(\bar{x}) \& x \geq \bar{x} \} \), which is the set of all solutions of the system described by (1) and (2).

Theorem 2.1 Let \( \hat{x} \) and \( S_j(\hat{x}_j) \) be defined as in Lemma 2.2 then:

(i) \( M^+(\bar{x}, \bar{x}) \neq \emptyset \) if and only if \( \hat{x} \in M^+(\bar{x}) \& \bar{x} \leq \hat{x} \).

(ii) If \( M^+(\bar{x}, \bar{x}) \neq \emptyset \), then \( \hat{x} \) is the maximum element of \( M^+(\bar{x}, \bar{x}) \).

(iii) Let \( M^+(\bar{x}, \bar{x}) \neq \emptyset \) and \( \bar{J} \subset J \). Let us set \( \tilde{x}_j = \hat{x}_j \) if \( j \in \bar{J} \), otherwise \( \tilde{x}_j = \bar{x}_j \). Then \( \bar{x} \in M^+(\bar{x}, \bar{x}) \Leftrightarrow \bigcup_{j \in \bar{J}} S_j(\tilde{x}_j) = I \)

Proof:
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(i) If $\hat{x} \in M^\pm(\overline{x}, \overline{\pi})$ & $x \leq \hat{x}$, from definition $M^\pm(x, \overline{\pi})$ we have $\hat{x} \in M^\pm(x, \overline{\pi})$, then $M^\pm(x, \overline{\pi}) \neq \emptyset$. If $M^\pm(x, \overline{\pi}) \neq \emptyset$ so that $M^\pm(\overline{\pi}) \neq \emptyset$ and from lemma 2.2, it is verified that $\hat{x} \in M^\pm(\overline{\pi})$ and $\hat{x}$ is the maximum element of $M^\pm(\overline{\pi})$, therefore $\hat{x} \geq \overline{x}$.

(ii) If $M^\pm(x, \overline{\pi}) \neq \emptyset$, so that $M^\pm(\overline{\pi}) \neq \emptyset$ and $M^\pm(x, \overline{\pi}) \subset M^\pm(\overline{\pi})$, and since $\hat{x}$ is the maximum element of $M^\pm(\overline{\pi})$, then $\hat{x}$ is the maximum element of $M^\pm(x, \overline{\pi})$.

(iii) Let $\tilde{x} \in M^\pm(x, \overline{\pi}) \Rightarrow \tilde{x} \in M^\pm(\overline{\pi})$, also from definition $\tilde{x}$ we have $\tilde{x}_j = \hat{x}_j$ for all $j \in J$, so that $S_j(\tilde{x}_j) \neq \emptyset \forall j \in J$. And $\tilde{x}_j < \hat{x}_j$ for all $j \in J \setminus \hat{J}$, so that $S_j(\tilde{x}_j) = \emptyset \forall j \in J \setminus \hat{J}$. Hence $\tilde{x} \in M^\pm(\overline{\pi})$, by lemma 2.2 we have $\bigcup_{j \in J} S_j(\tilde{x}_j) = I$.

Let $\bigcup_{j \in \hat{J}} S_j(\tilde{x}_j) = I$, since $\hat{J} \subseteq J$ we have $\bigcup_{j \in J} S_j(\tilde{x}_j) = I$. By lemma 2.2, $\tilde{x} \in M^\pm(\overline{\pi})$ also we have $\tilde{x} \geq \overline{x}$ therefore $\tilde{x} \in M^\pm(x, \overline{\pi})$.

3. Optimization Problems Under One-sided \((\max, \min)\)-Linear Equality Constraints

In this section we will solve the following optimization problem:

$$f(x) \equiv \max_{j \in J} f_j(x_j) \rightarrow \min \tag{3}$$

subject to

$$x \in M^\pm(x, \overline{\pi}) \tag{4}$$

We assume further that $f_j : R \rightarrow R$ are continuous and monotone functions (i.e. increasing or decreasing). $M^\pm(x, \overline{\pi})$ denotes the set of all feasible solutions of the system described by (1) and (2) and assuming that $M^\pm(x, \overline{\pi}) \neq \emptyset$ (note that the emptiness of the set $M^\pm(x, \overline{\pi})$ can be verified using the considerations of the preceding section). Let $J^* \equiv \{ j \mid f_j \text{ decreasing function} \}$ so that

$$\min_{x_j \in [\tilde{x}_j, \hat{x}_j]} f_j(x_j) = f_j(\hat{x}_j), \ \forall j \in J^*.$$ 

Then we can propose an algorithm for solving problem (3) and (4) under the assumption that $M^\pm(x, \overline{\pi}) \neq \emptyset$ which means $\bigcup_{j \in J} S_j(\tilde{x}_j) = I$, i.e. for finding an optimal solution $x^{opt}$ of problem (3) and (4).

Algorithm 3.1 We will provide algorithm, which summarizes the above discussion and finds an optimal solution $x^{opt}$ of problem (3) and (4), where $f_j(x_j)$ are continuous and monotone functions.

0 Input $I$, $J$, $\overline{x}$, $\overline{\pi}$, $a_{ij}$ and $b_i$ for all $i \in I$ and $j \in J$;

1 Find $\tilde{x}$, and set $\tilde{x} = \hat{x}$,

2 Find $J^* \equiv \{ j \mid f_j \text{ decreasing function} \}$

3 $F = \{ p \mid \max_{j \in J} f_j(\tilde{x}_p) = f_p(\tilde{x}_p) \}$

4 If $F \cap J^* \neq \emptyset$, then $x^{opt} = \tilde{x}$, Stop.

5 Set $y_p = x_p \ \forall \ p \in F$, & $y_j = \tilde{x}_j$, otherwise

6 If $\bigcup_{j \in J} S_j(y_j) = I$, set $\tilde{x} = y$ go to 3

7 $x^{opt} = \tilde{x}$, Stop.

We will illustrate the performance of this algorithm by the following numerical example.
**Example 3.1** Consider the optimization problem (3), where \( f_j(x_j) \ \forall \ j \in J \) are continuous and monotone functions in the form \( f_j(x_j) = c_j \times x_j + d_j \),

\[
C = \begin{bmatrix} -0.2057 & 4.8742 & 2.8848 & 0.9861 & 1.7238 & 1.1737 & -3.3199 \end{bmatrix}
\]

and

\[
D = \begin{bmatrix} 1.4510 & 1.5346 & -3.6121 & -0.9143 & -2.0145 & 1.9373 & -4.8467 \end{bmatrix}
\]

subject to \( x \in M = (x, \bar{x}) \), where the set \( M = (x, \bar{x}) \) is given by the system (1) and (2) where \( J = \{1, 2, \ldots, 7\} \), \( I = \{1, 2, \ldots, 6\} \), \( x_j = 0 \ \forall \ j \in J \) and \( \bar{x}_j = 10 \ \forall \ j \in J \) and consider the system (1) of equations where \( a_{ij} \) \& \( b_i \) \ \forall \ i \in I \) and \( j \in J \) are given by the matrix \( A \) and vector \( B \) as follows:

\[
A = \begin{bmatrix}
2.0115 & 6.3539 & 4.4317 & 7.7452 & 0.6465 & 9.4098 & 1.3576 \\
8.5668 & 5.8310 & 2.5146 & 8.7804 & 3.7709 & 4.4770 & 2.3007 \\
\end{bmatrix}
\]

and

\[
B^T = \begin{bmatrix}
\end{bmatrix}
\]

By the method in section 2 we get \( \hat{x} \), which is the maximum element of \( M = (x, \bar{x}) \), as follows:

\[
\hat{x} = (6.5712, 6.1221, 6.1221, 6.3539, 6.4355, 6.3539, 7.0955)
\]

By using algorithm 3.1 after five iterations we find that if we set \( x_4 = 0 \) the third equation of the system (1) does not satisfy, therefore ALGORITHM 1 go to step 8 and take

\[
x^{opt} = \hat{x} = (6.5712, 0, 0, 6.3539, 0, 0, 7.0955)
\]

and stop. We obtained the optimal value for the objective function \( f(x^{opt}) = 5.3510 \). We can easily verify that \( x^{opt} \) is a feasible solution.

**Remark 3.1** By reference to the lemma 2.2 and theorem 2.1 it is not difficult to note that the maximum number of arithmetic or logic operations in any step to get \( \hat{x} \) can not exceed \( n \times m \) operations. This will happen when we calculate \( x_j^{(i)} \), \( \forall \ i \in I \) \& \( j \in J \). Also from the above example we can remark that the maximum number of operations in each step in any iterations from algorithm 3.1 is less than or equal to the number of variables \( n \) and the maximum number of iterations from step 3 to step 6 of this algorithm can not exceed \( n \). Therefore the computational complexity of the algorithm 3.1 is \( O(max(n^2, n \times m)) \).

In what follows let us modify algorithm 3.1, to be suitable to find an optimal solution for any general continuous functions \( f_j(x_j) \) as follows:

**Algorithm 3.2** We will provide algorithm, which summarizes the above discussion and find an optimal solution \( x^{opt} \) of problem (3) and (4), where \( f_j(x_j) \) are general continuous functions:

1. **Input** \( I, J, x, \bar{x}, a_{ij} \) \& \( b_i \) for all \( i \in I \) \& \( j \in J \);
2. **Find** \( \hat{x} \), \text{ and set } \hat{x} = \hat{x};
3. **Find** \( \min_{x_j \in [\underline{x}_j, \bar{x}_j]} f_j(x_j) = f_j(x^*_j), \ \forall \ j \in J \);
4. **Set** \( J^* \equiv \{ j \mid f_j(\hat{x}_j) = f_j(x^*_j) \} \)
5. **Set** \( F = \{ p \mid \max_{j \in J} f_j(\hat{x}_j) = f_p(\hat{x}_p) \} \)
6. **If** \( F \cap J^* \neq \emptyset \), then \( x^{opt} = \hat{x} \), Stop.
Set $y_p = x_p^* \forall \ p \in F$, \& \ $y_j = \tilde{x}_j$, \ otherwise

If $\bigcup_{j \in J} S_j(y_j) = I$, \ set \ $\tilde{x} = y$ \ go to \(7\)

$x^{opt} = \tilde{x}$, \ Stop.

We will illustrate the performance of this algorithm by the following numerical examples.

**Example 3.2** Consider the optimization problem (3), where $f_j(x_j) \ \forall \ j \in J$ are continuous functions given in the following form $f_j(x_j) \equiv (x_j - \xi_j)^2$.

$\xi = (3.3529, 1.4656, 5.6084, 5.6532, 6.1536, 6.5893)$

subject to $x \in M^+(x, \overline{x})$, where the set $M^+(x, \overline{x})$ is given by the system (1) and (2) where $J = \{1, 2, \ldots, 6\}$, $I = \{1, 2, \ldots, 6\}$, $x_j = 0 \ \forall \ j \in J$ and $\overline{x} = 10 \ \forall \ j \in J$ and consider the system (1) of equations where $a_{ij}$ & $b_i \ \forall \ i \in I$ and $j \in J$ are given by the matrix $A$ and vector $B$ as follows:

$$A = \begin{pmatrix}
3.6940 & 0.8740 & 0.5518 & 4.6963 & 2.1230 & 1.4673 \\
1.9585 & 8.3470 & 5.8150 & 8.5545 & 8.9532 & 8.7031 \\
1.3207 & 8.9610 & 1.5718 & 3.7155 & 0.1555 & 4.3611 \\
1.1172 & 3.6992 & 7.5108 & 4.7686 & 4.4845 & 4.3301
\end{pmatrix}$$

and

$$B^T = [4.0195 \ 7.2296 \ 4.2766 \ 6.6594 \ 4.1969 \ 6.9874]$$

By the method in section 2 we get $\hat{x}$, which is the maximum element of $M^+(x, \overline{x})$, as follows:

$\hat{x} = (6.6594, 4.1969, 6.9874, 4.0195, 7.2296, 4.2766)$

By using algorithm 3.2 after three iterations we find that $x^{opt} = (3.3297, 1.4689, 6.9874, 4.0195, 7.2296, 4.2766)$.

Here we find the algorithm 3.2 stop in step 5 since the active variable in iteration 3 is $x_6$, and at the same time the objective function has the minimum value in this value of $x_6$, so that the algorithm 3.2 stop. Then we obtained the optimal value of the objective function, $f(x^{opt}) = 5.3488$.

It is easy to verify that $x^{opt}$ is a feasible solution.

**Example 3.3** Consider the optimization problem (3), where $f_j(x_j) \ \forall \ j \in J$ are continuous functions given in the following form $f_j(x_j) \equiv |(x_j - \xi_j)(x_j - h_j)|$,

where

$\xi = (3.3529, 1.4656, 5.6084, 5.6532, 6.1536, 6.5893)$

and

$h = (0.7399, -0.1385, -4.1585, 1.1625, -2.1088, 1.2852)$

subject to $x \in M^+(x, \overline{x})$, where the set $M^+(x, \overline{x})$ is given in the same way as in example 3.2

By the method in section 2 we get $\hat{x}$, which is the maximum element of $M^+(x, \overline{x})$, as follows:

$\hat{x} = (6.6594, 4.1969, 6.9874, 4.0195, 7.2296, 4.2766)$

By using algorithm 3.2 we find:

**Iteration 1:**

1. $\hat{x} = (6.6594, 4.1969, 6.9874, 4.0195, 7.2296, 4.2766)$;
2. $x^* = 0.7325, 1.4689, 5.5899, 1.1656, 6.1452, 1.2830$;
3. $\mathcal{F} = \emptyset$;
4. $\mathcal{G} = \{1\}$;
5. $f(\hat{x}) = 19.5728$;
6. $y = (0.7325, 4.1969, 6.9874, 4.0195, 7.2296, 4.2766)$;
7. $\bigcup_{j \in J} S_j(y_j) = I$;
\[ \tilde{x} = (0.7325, 4.1969, 6.9874, 4.0195, 7.2296, 4.2766). \]

**Iteration 2:**

5. \( J^* = \{ 1 \}; \)
6. \( F = \{ 3 \}; \)
7. \( f(\tilde{x}) = 15.3692; \)
8. \( y = (0.7325, 4.1969, 5.5899, 4.0195, 7.2296, 4.2766); \)
9. \( \bigcup_{j \in J} S_j(y_j) = \{ 1, 2, 3, 5 \} \neq I; \)
10. \( x^{opt} = (0.7325, 4.1969, 6.9874, 4.0195, 7.2296, 4.2766), \text{STOP}. \)

In Iteration 2 algorithm 3.2 stops since the fourth and sixth equations in the system of equations (1) can not be verify if we change the value of the variable \( y_3 \) to be \( y_3 = (5.5899) \). If it is necessary to change this value of the variable \( y_3 \) to minimize the objective function so that our exhortation to decision-maker, it must be changed both the capacities of the ways \( a_{45} \) and \( a_{65} \) to be \( a_{45} = 6.6594 \) and \( a_{65} = 6.9874 \) in order to maintain the verification of the system of equations (1). In this case we can complete the operation to minimize the objective function, after Iteration 4 we obtained \( x^{opt} = (0.7325, 1.4689, 5.5899, 4.0195, 7.2296, 4.2766); \) so that the optimal value \( f(x^{opt}) = 10.0481 \), we can easily verify that \( x^{opt} \) is a feasible solution.

**Remark 3.2** Step [2] in algorithm 3.2 depends on the method, which finds the minimum for each function \( f_j(x_j) \) in the interval \([\bar{x}_j, \tilde{x}_j]\), but this appears in the first iteration only and only once. Also from the above examples we can remark that the maximum number of operations in each step in any iterations from algorithm 3.2 is less than or equal to the number of variables \( n \) and the maximum number of iterations from step [3] to step [7] of this algorithm can not exceed \( n \). Therefore the computational complexity of the algorithm 3.2 is given by \( \max \left\{ O(\max(n^2, n \times m)), \bar{O} \right\} \), where \( \bar{O} \) is complexity of the method, which finds the minimum for each function \( f_j(x_j) \) in the interval \([\bar{x}_j, \tilde{x}_j]\).

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**References**


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