Growth of Errors in Weather Prediction with Use of Low-dimensional Atmospheric Model

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Abstract. The paper studies low-dimensional atmospheric model introduced by Lorenz in 1996. The relevance and common properties of the model are discussed with regard to chaotic behavior identification under selected conditions. For this purpose, Lyapunov exponents are estimated and compared with the ensemble prediction approach. The comparison discovers different initial error growth of these methods. The consequences of the results for this and for more complex models are discussed.

1. Introduction

If we want to construct a operation global circulation model we try to duplicate the atmosphere and its surroundings as closely as it is possible. We are, of course limited by our knowledge and available computing facilities. In this article, we try to simulate an atmospheric behavior from a different view. We follow the work of Lorenz [Lorenz, 1996]. In 1996, he introduced the very simple model trying to represent an one-dimensional atmosphere.

Note that Lorenz also discovered chaotic behavior of the atmosphere [Lorenz, 1963]. That implies an exponential growth of initial error. The Growth rate of the error is governed by Lyapunov exponent. For a weather forecast, the rate of initial error growth is very important. In this paper we focus on the question whether the Lyapunov exponent describe it well or not. To do so we compared it with the ensemble prediction approach.

Before we do that, we discuss the properties and relevance of the Lorenz’s 1996 model which we use as the model of atmosphere.

2. The model

Lorenz [Lorenz, 1996] introduced the model with N variables $X_1, ..., X_N$ with governing equations

$$\frac{dX_n}{dt} = -X_{n-2}X_{n-1} + X_{n-1}X_{n+1} - X_n + F,$$  \hspace{1cm} (2.1)

$X_{n-2}, X_{n-1}, X_n, X_{n+1}$ are a closely unspecified scalar meteorological quantity, $F$ is a constant with meaning of some forcing and $t$ is time. The index $n$ is cyclic so that $X_{n-2} = X_{n-N} = X_n$ and the variables can be viewed as existing around a circle. The nonlinear terms of equation (2.1) simulate advection. The linear terms represent the mechanical and thermal dissipation. The constant terms simulate an external forcing. The model quantitatively, to a certain extent, describes weather systems, but the equations (2.1) cannot be derived from any atmospheric dynamic equations. The goal was to formulate the simplest possible set of N dissipative chaotically behaving differential equations that share some properties with the “real” atmosphere. In Lorenz and Emanuel [1998] and Lorenz [2005] the reasoning for usability of such model is discussed in more detail.

Let $r = \sum_{n=1}^{N} X_n$ and $s^2 = \sum_{n=1}^{N} X_n^2$. Total energy is defined as $s^2/2$ and substitutes it into the model equations (2.1) lead to:

$$\frac{d(s^2/2)}{dt} = -s^2 + Fr.$$  \hspace{1cm} (2.2)

The quadratic terms in eq. (2.1) have canceled. According to Lorenz [2005], it is common for many atmospheric models, that an advection term does not add or remove an energy.

Let define $\overline{X}$ and $\overline{X^2}$ like an averages over all values of $n$ and over a long enough time in order to have the average time derivatives negligibly small. From the equation (2.2) follows that

$$\overline{X^2} = F\overline{X},$$  \hspace{1cm} (2.3)

and
\[ \bar{X}^2 - \bar{X}'^2 = \bar{X} \left( F - \bar{X} \right). \]  

(2.4)

The variance \( \sigma^2 = \bar{X}^2 - \bar{X}'^2 \) is nonnegative. From the variance and the equation (2.4) follow that the mean \( \bar{X} \in [0, F] \) and the standard deviation \( \sigma \in [0, F/2] \).

Let study stability of the model (2.1). If \( X_n = F \) for each \( n \), than \( \bar{X} = F \), \( \sigma = 0 \) — such steady solution is stable. We obey the perturbation method, which is explained for example in Holton [2004] in a meteorological context. Let’s define \( X_n = F + x_n \), where \( x_n \) are a small perturbations about the steady solution

\[ x_n = \exp \left( ht \right) \cos \left( kn - mt \right), \]

(2.5)

\( h \) decides whenever the perturbation will grow or will not, \( k \) is a wavenumber and \( m \) represent an angular velocity. The substitution \( X_n = F + x_n \) into the eqn. (2.1) and the supposition that \( |x_n| / F| \leq 1 \) (the terms \( x_n x_{n-1} \) and \( x_{n-1} x_{n+1} \) can be neglected) lead to:

\[ \frac{dx_n}{dt} = F (x_{n+1} - x_{n-1}) - x_n. \]

(2.6)

With our form (2.5) of \( x_n \), the equation (2.6) leads to

\[ \sin \left( kn - mt \right) \left[ -m + F \sin \left( kn \right) + F \sin \left( 2k \right) \right] = \cos \left( kn - mt \right) \left[ -h + F \cos \left( k \right) - F \cos \left( 2k \right) \right]. \]

(2.7)

It is possible only if

\[ h = F \left( \cos \left( k \right) - \cos \left( 2k \right) \right) - 1, \]

(2.8)

\[ m = -F \left( \sin \left( k \right) + \sin \left( 2k \right) \right). \]

(2.9)

If \( h > 0 \), then a small perturbation will grow, and the solution will be unstable. If \( h < 0 \), then the steady state will be stable. Let discuss the equation (2.8). For our purpose we use the simplest relevant system with \( N = 4 \) and \( F > 0 \). The zonal wavenumber \( k \) must be an integer divisor of \( N \). Table 2.1 shows the wavelengths \( \lambda \), the wavenumbers \( k = 2\pi / \lambda \) and the values of \( \cos \left( k \right) - \cos \left( 2k \right) \). The unstable steady solution is for \( \lambda = 4 \) and \( F > 1 \).

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>( 2\pi )</td>
<td>( \pi )</td>
<td>( 2\pi / 3 )</td>
<td>( \pi / 2 )</td>
</tr>
<tr>
<td>( \cos k - \cos 2k )</td>
<td>0</td>
<td>-2</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The phase velocity for the wavelength \( \lambda = 4 \) and \( F = 1 \) is \( c_p = m / k = -0.64 \). Such wave will drift westward. Its group velocity is \( c_g = \partial m / \partial k = -\left( \cos \left( k \right) + 2 \cos \left( 2k \right) \right) / 2 \) — this implies the eastward propagation of the regions of activity. This is common, for example, with Rossby waves and it is the important consideration to use the model. On the other hand we cannot use a model where only a negative phase velocity and a positive group velocity for a different wave number \( k \) can occur. Figure 2.1, with \( F = 1 \), shows that all the propagations are possible.

3. The Numerical solution and chaotic behavior of the model

The suitable numerical integration method of eqn. (2.1) was studied Bednar [2010]. Results lead to the fourth-order Runge-Kutta scheme and system is numerically stable with the step \( h = 0.005 \) units. If we consider 1 unit = 5 days, the step is 0.6 hour.

To demonstrate the information about the model from the chapter two by using numerical method, we firstly introduce the points of static equilibrium \( \bar{X}_e \), which satisfy:

\[ \frac{dX_e}{dt} \bigg|_{X_e} = -X_{e+2} X_{e+1} + X_{e+2} X_{e+1} - X_{e+1} + X_e + F = 0. \]

(3.1)
The physical relevant solution of the equations 3.1 is then expected $X_F = F$.

We can then decide on stability and instability by looking at the eigenvalues of the system’s Jacobian matrix $J$, computed at the point of static equilibrium $X_F = F$:

$$J = \begin{pmatrix}
-1 & 0 & -F & F \\
F & -1 & 0 & F \\
-F & F & -1 & 0 \\
0 & -F & F & -1
\end{pmatrix}. \quad (3.2)$$

The table 3.1 shows the signs of the real (third and eighth column) and the imaginary part (fourth and ninth column) of the eigenvalues on the interval $(0;30)$ of $F$. The figure confirms results from the chapter two.

<table>
<thead>
<tr>
<th>interval $F$</th>
<th>eigenvalues</th>
<th>sign R</th>
<th>sign I</th>
<th>stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(0,1&gt;$</td>
<td>1</td>
<td>–</td>
<td>0</td>
<td>yes</td>
</tr>
<tr>
<td>2</td>
<td>–</td>
<td>+</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>–</td>
<td>–</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>–</td>
<td>0</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(1;30&gt;$</td>
<td>1</td>
<td>–</td>
<td>0</td>
<td>no</td>
</tr>
<tr>
<td>2</td>
<td>+</td>
<td>+</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>+</td>
<td>–</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>–</td>
<td>0</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Edward Lorenz in sixties coincidentally discovered chaotic behavior of the atmosphere. By the monograph of Sprott [2003]: “Chaos is the aperiodic, long-term behavior of a bounded, deterministic system that exhibits sensitive dependence on initial conditions.”

The forcing parameter $F$ plays leading role for the model behavior. To get rough idea about a possible appearance of chaotic behavior in dependence on the parameter $F$ we constructed the bifurcation diagram (Fig. 3.1) for $X_1$ variable; analogical behavior is expected for other variables. The bifurcation diagram shows, that our range of interest is for $F > 10$, because from that value the system looks aperiodically.

![Bifurcation diagram](image)
Sensitive dependence on initial conditions means an exponential growth of two infinitesimally close initial conditions. Time evolution, $\Delta d(t)$, of the initial conditions of, $\Delta d_x$, subjects the well-known relation:

$$\Delta d(t) = \Delta d_x e^{\lambda t}$$

where $\lambda$ denotes the largest Lyapunov exponent.

The distance, $\Delta d(t)$, is measured in a phase space whose coordinates are the variables of a system. $N$ dimensional system has $N$ Lyapunov exponents. So if we talk more generally, we observe an infinitesimal sphere in a phase space which surface is filled with the initial states in $t=0$. By time evolution, the sphere suffers deformation to ellipsoid or a shape that is more complicated. The long-term averages of the deformation are linked by eqn. (3.3) to the Lyapunov exponents. One may ask about the difference between the Lyapunov exponents and the eigenvalues, because they both are determined from the Jacobian matrix of the system and assuming local linearization of the dynamic. The difference shows table 3.2.

**Table 3.2** The difference between the Eigenvalues and the Lyapunov exponents.

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Eigenvalue</th>
<th>Lyapunov exponent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>Local</td>
<td>Global</td>
</tr>
<tr>
<td>Number</td>
<td>Complex</td>
<td>Real</td>
</tr>
<tr>
<td>Constant</td>
<td>Average</td>
<td></td>
</tr>
</tbody>
</table>

Time evolution of the system (eqn. (2.1) or generally any dynamical system $d\tilde{X}/dt = f(\tilde{X})$) is in a phase space represented by the trajectory. Let $\tilde{e}(t) = \delta\tilde{X}(t)$ be small change from that trajectory. The change evolves as:

$$\frac{d\tilde{e}}{dt} = \tilde{J}\tilde{e},$$

where $\tilde{J} = \frac{\partial f}{\partial X}$ is Jacobian matrix.

Every linear eqn. (3.4) represents the evolution of the orthogonal Lyapunov vector (one from $N$) which represents evolution of the initial sphere. Ratio $e_n(t)/e_n(0)$ of the Lyapunov vector in $n$ direction substituted into the eqn. (3.3) gives:

$$\frac{e_n(t)}{e_n(0)} = e^{\lambda_{n,local} t},$$

$$\lambda_{n,local} = \frac{1}{t} \ln \frac{e_n(t)}{e_n(0)}.$$  

Time averaging of the last eqn. leads to the global quantity ($n$-th Lyapunov exponent):

$$\lambda_n = \lim_{t \to \infty} \frac{1}{t} \ln \frac{e_n(t)}{e_n(0)}.$$  

A bounded dynamical system with a positive Lyapunov exponent is chaotic, and the exponent describes the average rate at which the predictability is lost. A bounded dissipative systems with continuous time $t$ are characterized by a negative value of sum of all the Lyapunov exponents.

It is not the goal of our article to find the value of parameter $F$ where chaos appears. We are more interested in certain Lyapunov exponents, especially those, which values are similar to the atmospheric ones. Because of the meteorological models are large-dimensional problems. It is not easy to find the exponents. Even though Lorenz [1996] mentions that ECMWF suggested the Lyapunov exponents between 2.1 and 2.4 for the errors in the 500 hPa height. The table 3.3 shows two parameters $F$ with the related exponents that we chose to use. The figure 3.2 presents the convergence of the largest Lyapunov exponents for those parameters of $F$.

The exponents in this figure satisfy rather eqn. (3.6) than (3.7). Because we are interest only in the growth of the initial error in the first few days and the exponents did not significantly vary with time for the later states and computational time is much longer than the region of interest, we can consider them as global.
Table 3.3 The spectrums of the Lyapunov exponents for the model.

<table>
<thead>
<tr>
<th>F</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\lambda_3$</th>
<th>$\lambda_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>2.4</td>
<td>0</td>
<td>-2</td>
<td>-4.2</td>
</tr>
<tr>
<td>16</td>
<td>2.18</td>
<td>0</td>
<td>-2</td>
<td>-4</td>
</tr>
</tbody>
</table>

Figure 3.2 Convergence of the largest Lyapunov exponent for $F = 18$ (left) and $F = 16$ (right).

4. The method

In this section we will compare the initial error growth governed by the Lyapunov exponent (method 1) with the estimations from the ensemble prediction (method 2). We will also discuss the accuracy of the method 1 for first few days of prediction.

Let introduce the method 2. We set the initial errors $e_{0n} = 0.001$, $n = 1, \ldots, 4$, which are same for all the variables. $X_{0n}$ are “observed” values and $X_{\omega} = X_{0n} + e_{0n}$ represent “real states.” The long term numerical integrations of eqn. (2.1) are done for $F = 16$, $F = 18$ (after transient period). The numerical integration is done for $K = 150$ steps, with the step size $h = 0.005$, that mean for 3.75 days. We have four variables, so we receive four sequences $X_{\omega}, \ldots, X_{\omega}$ and $X'_{\omega}, \ldots, X'_{\omega}$. We evaluate $e_{n} = X_{\omega} - X_{\omega}$ for all $n$ and $k = 1, \ldots, 150$. We repeat such sequence $M = 1000$-times. The new initial values are taken as the last ones from previous runs, i.e. $X_{0n,m} = X_{0n,m-1}, m = 1, \ldots, M$. In every step, we calculate the variance $e_{n,\omega}^2 = \frac{1}{4} \left( e_{1n}^2 + e_{1n}^2 + e_{1n}^2 + e_{1n}^2 \right)$ and the average variance for all runs at the certain step $E_{s}^2 = \frac{1}{M} \left( e_{1n}^2 + e_{1n}^2 + \ldots + e_{1n, M}^2 \right)$. We plot $E_{s}, k = 1, \ldots, 150$ against the method 1 ($\Delta d_i = 0.001 \cdot e_{10}^2$) for $F = 16$ and $F = 18$ and also $\ln \left( \frac{E_{s}}{E_{0}} \right)$ and $\ln \left( \Delta d_i / 0.001 \right) = \lambda_i \cdot h \cdot k$ for $F = 16$ and $F = 18$. Results are shown in Figs. 4.1.

We can divide these results into four classes presented in table 4.1. Note that the Lyapunov exponent method overestimate the error for the first 30 hours, while later underestimates it. Furthermore, after 90 hours “real” errors are almost twice greater than in Lyapunov exponent approach. Special attention can be devoted to the interesting result, that the errors measured by ensemble method is decreasing during first 10 hours, and after approximately 15 hours is firstly large than the initial perturbation. Smagorinsky [1969] has also described such error behavior in relation to the numerical weather prediction models. Both methods (1 and 2) show the same error after approx. 30 hours.

To explain the different initial error growth of the methods we have to remind the definition of a Lyapunov exponent as a long-term average characteristic. Lorenz [1996] than agree, that for firs few days the error growth differs from the one establishing from a Lyapunov exponent. That is true not only for a low dimensional model, but also for a global circulation models.

Lorenz [1996] also pointed out that the “real” value of the error evaluated from a low dimensional models is twice the size of the error calculated from Lyapunov exponent for time between third and fourth day and that the results, which come from a low dimensional model, are similar to those of global circulation models.
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Figure 4.1 Comparison of $E_i$ (black) and $\Delta d_{i}$ (red) for $F = 16$ (first) and $F = 18$ (second) and comparison of $\ln \left( \frac{E_i}{E_0} \right)$ (black) and $\ln \left( \frac{\Delta d_{i}}{0.001} \right) = \lambda \cdot h \cdot k$ (red) for $F = 16$ (third) and $F = 18$ (fourth).

Table 4.1: Results of figure 4.1.

<table>
<thead>
<tr>
<th>Interval [hour]</th>
<th>method 1</th>
<th>method 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>the error is growing</td>
<td>the error is bigger than initial error</td>
</tr>
<tr>
<td>(0; 10)</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>(10; 15)</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>(15; 30)</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>(30; 90)</td>
<td>yes</td>
<td>no</td>
</tr>
</tbody>
</table>

5. Conclusions

Our experiments compared two approaches to the initial error growth in the low dimensional atmospheric model. The results showed that the ensemble and the Lyapunov exponent concepts are leading to the different error estimations. Chaos theory familiar Lyapunov exponents cannot be relevant for description of the initial error growth during all integration time. Ensemble prediction method discovered relatively very low error for first day of prediction, while after 90 hours the error has double size in comparison with the error of the Lyapunov exponents approach. All these results are in good agreements with the Lorenz conclusions [Lorenz, 1996]. Importance of the issues is based on the discussion of the model in first part of the article, where the authors have tried to explain the legitimacy of the low dimensional model equations in revealing and interpretations of more complex models.

References


