WDS'11 Proceedings of Contributed Papers, Part I, 67–71, 2011. ISBN 978-80-7378-184-2 © MATFYZPRESS

Comparison of Clenshaw-Curtis and Gauss Quadrature

M. Novelinková

Charles University, Faculty of Mathematics and Physics, Prague, Czech Republic.

Abstract. In the present work, Gauss and Clenshaw-Curtis quadrature formulas are compared. It is well known, that Gauss quadrature converges for every continuous function f and has a factor-of-2 advantage in efficiency for finite n ($(n+1)$-point scheme integrates exactly polynomials of degree $2n+1$) On the other hand, Clenshaw-Curtis scheme integrates exactly polynomials of degree at most n, but converges also for every continuous function f. This scheme does not turn out to be half as efficient as the Gauss formula for most of the integrands, both quadratures reach almost the same accuracy. Moreover, using the fast Fourier transformation, the Clenshaw-Curtis scheme can be implemented in $\mathcal{O}(n \log n)$ operations, what makes the scheme more efficient. Gauss quadrature nodes and weights can be evaluated in $\mathcal{O}(n^2)$ operations solving a tridiagonal eigenvalue problem. However, there are some problems (boundary elements integrals) where the Gauss quadrature should continue to be preferred quadrature rule.

Introduction

In this article we will consider the following problem. We are given a continuous function f on the closed interval $[-1, 1]$ and we seek to approximate the integral

$$I = I(f) = \int_{-1}^{1} f(x)\mathrm{d}x$$

by sums

$$I_n = I_n(f) = \sum_{k=0}^{n} w_k f(x_k)$$

for various integers n, where the nodes x_k depend on n but not on the function f itself. The weights are defined uniquely by the property that I_n is wanted to be *interpolatory quadrature*, what means that it integrates exactly polynomials of degree at most n.

The most important approach to such a problem is through the automatic quadrature scheme, what is a set of quadrature formulas, each with its own error estimate. According to the specified tolerence the one of these formulas is chosen, without requiring an excessive number of values of the integrand. The quadrature formulas and the error estimate must be possible to evaluate using only values of the integrand in the interval of integration. Usually, the formulas are applied in sequence so it is very suitable that function values required by one formula can be used by the later ones.

There are several methods how to built a quadrature formula. The well known Newton-Cotes formulas are defined by taking the nodes to be equally spaced from -1 to 1. The properties of such schemes for $n \to \infty$ are very bad, some of the weights are negative and the formulas do not converge for a general continuous integrand f. They converge only if the function f is analytic in large region surrounding the interval of integration.

Another very significant example is the Gauss quadrature formula, which is defined by choosing the nodes optimally in the sense of maximizing the degree of polynomials that (1) can integrate exactly. Since there are $n+1$ nodes, the attainable degree is $n+1$ order higher. Thus the $(n+1)$-point Gauss formula integrates exactly polynomials of degree $2n+1$, what makes it the most accurate.

Clenshaw-Curtis scheme

There are two ways how to describe the idea behind the Clenshaw-Curtis scheme. Firstly, we construct the interpolatory polynomials that have the same values as the integrand in the zeros of the Tchebyshev polynomials. We observe that a sequence of such polynomials converges to the function almost everywhere for the piecewise continuous function (unlike the interpolatory polynomials agreeing at equidistant nodes). That means that the function $f(x)$ will be firstly approximated by an interpolating polynomial agreeing with it in the Tchebyshev points, and then this polynomial will be integrated.
The second, equivalent, way is to substituate the variable of integration

$$\int_{-1}^{1} f(x)\mathrm{d}x = \int_{0}^{\pi} f(\cos\theta)\sin\theta\mathrm{d}\theta. \tag{1}$$

If we knew the cosine transform of the function f

$$f(\cos\theta) = F(\theta) = \sum_{n=0}^{\infty} A_n \cos(n\theta) \tag{2}$$

then the integral (1) could be rewritten as follows

$$\int_{0}^{\pi} \sum_{n=0}^{\infty} A_n \cos(n\theta)\sin\theta\mathrm{d}\theta = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \frac{2A_n}{1-n^2} \tag{3}$$

The cosine transform of the function f is, of course, not known, but if we compute the disctrete finite cosine transform of $F(\theta)$ sampled at equidistant points $\theta = \frac{\pi s}{N}$, $s = 0, 1, \ldots, N$, then we obtain[1]

$$a_n = 2\sum_{s=0}^{N}{}'' F\left(\frac{\pi s}{N}\right)\cos\left(\frac{\pi s n}{N}\right). \tag{4}$$

Then the inverse formula gives

$$F\left(\frac{\pi s}{N}\right) = \frac{1}{N}\sum_{n=0}^{N}{}'' a_n \cos\left(\frac{\pi s n}{N}\right). \tag{5}$$

Thus we can use this formula and approximate the integrand

$$F(\theta) \approx \sum_{n=0}^{N}{}'' \left(\frac{a_n}{N}\right)\cos(n\theta). \tag{6}$$

Finally, we obtain the Clenshaw-Curtis quadrature formula

$$\int_{0}^{\pi} f(\cos\theta)\sin\theta\mathrm{d}\theta \approx \sum_{\substack{n=0 \\ n \text{ even}}}^{N}{}'' \left(\frac{a_n}{N}\right)\frac{2}{1-n^2}. \tag{7}$$

The expansion to the cosine series exists and converges for every function f continuous with bounded variation. Many properties now follow from classical Fourier series results such as Parseval's theorem or the Nyquist sampling theorem. It was proved [*Imhof*, 1963] that the

[1]in these and later formulas the symbol $\sum''$ means that the first and the last terms of the sum have half weight

weights a_n are positive, the scheme integrates polynomials of degree n exactly and converges for all continuous function f (for details see [*Novelinkova*, 2010]). The high cost of the cosine transform was a serious drawback in using this type of quadrature formula. Furthermore, the conventional computation of the cosine transform using the recurrence relation [*Engels*, 1980] is numerically unstable, particularly at the low frequencies. Also in case the automatic scheme requires refinement of the sampling, massive storage is needed to save the integrand values after the cosine transformation is computed.

In the paper [*Gentleman*, 1972] a modification of the fast Fourier transform was introduced, what overcomes all the problems mentioned above, it is very resistant to rounding errors and can be implemented in $O(n \log n)$ [*Waldvogel*, 2006]. In order not to waste integrand values already obtained, the new choice of the number of nodes n should be a multiple of the previous one, usually this is taken as $n = 2^p$ [*Gentleman*, 1972].

There are many error estimates for this quadrature scheme proposed, for example see [*Clenshaw and Curtis*, 1960], [*O'Hara and Smith*, 1967], [*Gentleman*, 1972], all of them can be easily calculated, what is very important for checking the accuracy obtained.

Comparison

As written in the introduction section, the Gauss quadrature is positive and also converges for every continuous function $f \in \mathcal{C}([-1,1])$. The nodes and weights of this formula can be evaluated in $\mathcal{O}(n^2)$ operations by solving a tridiagonal eigenvalue problem, as was proved in [*Golub and Welsch*, 1969].

The basic comparison is straightforward, the Gauss quadrature appear to have factor-of-2 advantage in efficiency for finite f, on the other hand, Clenshaw-Curtis scheme is easier and faster to implement.

However, the numerical comparison of the Clenshaw-Curtis and the Gauss quadrature schemes can reveal suprising results [*O'Hara and Smith*, 1967], [*Trefethen*, 2008]]: Clenshaw-Curtis scheme reaches almost the same accuracy for most of the integrands. The Gauss quadrature significantly outperforms the Clenshaw-Curtis quadrature only for functions analytic in a sizable neighborhood of $[-1,1]$. For such functions, the convergence of both methods is very fast. Thus Clenshaw-Curtis quadrature essentially never requires many more evaluations than Gauss to converge to a prescribed accuracy. Even Clenshaw and Curtis themselves recorded the same effect [*Clenshaw and Curtis*, 1960]. There were several error estimations stated in [*O'Hara and Smith*, 1972] and [*Trefethen*, 2008] that explains the practical equivalence of these two quadrature rules.

For the theoretical explanation of such a behavior we will introduce following definitions. Given $f \in \mathcal{C}([-1,1])$, and $n \geq 0$, let p_n^* be the unique best approximation to f on $[-1,1]$ of degree $\leq n$ with respect to the norm $||.|| = ||.||_\infty$ and define $E_n^* = ||f - p_n^*||$. For a quadrature scheme with nonnegative weights and for any $f \in \mathcal{C}([-1,1])$ the following statement holds

$$|I - I_n| \leq 4E_n^*, \tag{8}$$

and $I_n \to I$ as $n \to \infty$. Since both of the quadrature considered are nonnegative, this statement can be applied and thus we have that if the best approximants to f converge rapidly as $n \to \infty$ then I_n will converge rapidly to I. As in [*Trefethen*, 2008] this will be combined with results of the approximation theory to the effect that if f is smooth, its best approximants converge rapidly. The results derived from the Tchebyshev series for a function $f \in \mathcal{C}([-1,1])$ will be considered.

Let us construct the Tchebyshev series for $f \in \mathcal{C}([-1,1])$ as it was done in [*Trefethen*, 2008].

Then we obtain[2]

$$f(x) = \sum_{j=0}^{\infty}{}' a_j T_j(x), \qquad a_j = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_j(x)}{\sqrt{1-x^2}} \mathrm{d}x, \tag{9}$$

where the $T_j(x) = \cos(j \cos^{-1} x)$ are the Tchebyshev polynomials of degree j. The equal sign in the first formula is justified under the mild condition that f is Dini-continuous, in which case the series converges uniformly to f.

In the work [*Trefethen*, 2008] is showed, that if f is smooth, its Tchebysheff coefficients decrease rapidly. Two smoothness conditions are considered: a kth derivative satisfying a condition related to bounded variation, or analyticity in a neighborhood of $[-1, 1]$. Several theorems bounding the coefficients of the Tchebyshev expansion are proved, and finally the following theorem is stated. Let the norm $\|.\|_T$ be the Tchebyshev-weighted 1-norm defined by

$$\|u\|_T = \|\frac{u'(x)}{\sqrt{1-x^2}}\|_1.$$

Theorem 1. *Let Clenshaw-Curtis and Gauss quadrature be applied to a function $f \in \mathcal{C}([-1,1])$. If $f, f', \dots, f^{(k-1)}$ are absolutely continuous on $[-1,1]$ and $||f^{(k)}||_T = V < \infty$ for some $k \geq 1$, then for all sufficiently large n*

$$|I - I_n| \leq \frac{32V}{15\pi k(2n+1-k)^k}. \tag{10}$$

"Sufficiently large n" means for the Clenshaw-Cutis scheme that $n > n_k$ for some n_k that depends on k but not f or V and for the Gauss quadrature $n \geq \frac{k}{2}$.

Let the norm $\|.\|_T$ be the Tchebyshev-weighted 1-norm defined by

$$\|u\|_T = \|\frac{u'(x)}{\sqrt{1-x^2}}\|_1.$$

The factor 2^{-k} in the error bound applies to Clenshaw-Curtis quadrature too. The crucial fact of the proof is that of aliasing. On the grid in $[0, 2\pi]$ of $2n$ equally spaced points $\theta_j = \pi j/n$, $0 \leq j \leq 2n-1$, the functions $\cos((n+p)\pi\theta_j)$ and $\cos((n-p)\pi\theta_j)$ are indistinguishable. Applying this fact to the variable $x = \cos\theta$ we obtain

Theorem 2. *For any integer p with $0 \leq p \leq n$*

$$T_{n+p}(x_j) = T_{n-p}(x_j) \tag{11}$$

on the Tchebyshev grid. Consequently for the Clenshaw-Curtis scheme

$$I_n(T_{n+p}) = I_n(T_{n-p}) = I(T_{n-p}) = \begin{cases} \frac{2}{1-(n-p)^2}, & \text{for } n \pm p \text{ is even} \\ 0, & \text{for } n \pm p \text{ is odd.} \end{cases} \tag{12}$$

The error in integrating T_{n+p} can be expressed

$$I(T_{n+p}) - I_n(T_{n+p}) = \begin{cases} \frac{8pn}{n^4 - 2(p^2+1)n^2 + (p^2-1)^2}, & \text{for } n \pm p \text{ is even} \\ 0, & \text{for } n \pm p \text{ is odd.} \end{cases} \tag{13}$$

[2]the symbol $\sum{}'$ indicates that the term for $j = 0$ is multiplied by $1/2$

Clenshaw-Curtis formula has essentially the same performance for most integrands and can be implemented effortlessly by the fast Fourier transformation. This scheme applied to the functions analytic in a sizable neighborhood of $[-1, 1]$ exhibits a curious phenomenon, that explains also why Gauss quadrature outperforms this scheme in these cases.
When the number of nodes in the integration rule increases, the error of the Clenshaw-Curtis quadrature rule does not decay to zero evenly but in two distinct stages. In the work [*Weideman and Trefethen*, 2007] was proved that for the number of nodes n less than a critical value, the error behaves like $\mathcal{O}(\rho^{-2n})$, where ρ is a constant greater than 1. For these values the accuracy of Clenshaw-Curtis scheme is almost indistinguishable from the one of Gauss quadrature. With higher amount of nodes, the error decreases at the rate $\mathcal{O}(\rho^{-n})$. It means, that initially Clenshaw-Curtis scheme converges about as fast as the Gauss rule. The point in which the convergence switches from one rate to another can be seen also in the error curve constructed for this scheme as it was done in [*Weideman and Trefethen*, 2007]. It was proved also, the critical value of n where the kink occurs depends on the position of the singularity if the integrand.
Even though, the practical equivalency of these two schemes was stated in several papers, there are problems in which Gauss quadrature scheme should stay the preferred integration rule (for example [*Elliott, Johnston and Johnston*, 2008]).

Conclusion

In this paper, we have briefly introduced Clenshaw-Curtis scheme which belongs to the class of interpolatory quadrature schemes. We stated some of the most important characteristics as well as the inconvenience which caused that this scheme was not that much in use it the past. The comparison with the well-known Gauss formula was examined from several points of view and the theoretical explanation was given. In many cases, these two schemes can be considered as equivalent, however there are some problems in which the equivalency can not be applied. The outlook for the future work is to focus on the sensitivity of the schemes discussed.

Acknowledgments. The work was supported by the grant SVV-2011-263316

References

Clenshaw, C. W. and Curtis, A. R., *A method for numerical integration on an automatic computer*, Numer. Math., 2, pp. 197-205, 1960.

Elliott, D., Johnston, B. M., Johnston, P. R., *Clenshaw-Curtis and Gauss-Legendre quadrature for certain boundary element integrals*, SIAM J. SCI. COMPUT., 31, pp.510-530, 2008.

Engels, H., *Numerical quadrature and cubature*, London, Academia Press, 1980.

Gentleman, W. M., *Implementing Clenshaw-Curtis quadrature. I*, Comm. ACM, 15, pp. 337-342, 1972.

Gentleman, W. M., *Implementing Clenshaw-Curtis quadrature. II*, Comm. ACM, 15, pp. 343-346, 1972.

Golub, G. H. and Welsch, J. H., *Calculation of Gauss quadrature rules*, Math. Comp., 23, pp. 221-230, 1969.

Imhof, J. P., *On the method for numerical integration of Clenshaw and Curtis*, Numer. Math., 5, pp. 138-141, 1963.

Novelinkova, M., *Non-interpolatory quadratures and refined interpolatory quadratures*, Diploma thesis, 2010.

O'Hara, H. and Smith, F. J., *Error estimation in the Clenshaw-Curtis quadrature formula*, Compu. J., 11, pp.213-219, 1968

Trefethen, L. N., *Is Gauss quadrature better than Clenshaw-Curtis?*, SIAM Rev., 50, pp.67-87, 2008.

Waldvogel, J., *Fast construction of the fejer and Clenshaw-Curtis quadrature rules*, BIT, 46, pp.195-202, 2006.

Weideman, J. A. C. and Trefethen, L. N., *The kink phenomenon in Fejér and Clenshaw-Curtis quadrature*, Numer. Math., 107, pp.707-727, 2007.