The Axisymmetric Limit of the Post-Newtonian Dedekind Ellipsoids

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Abstract. We discuss a generalization to the post-Newtonian approximation for the Dedekind ellipsoids considered by Chandrasekhar and Elbert, [1974, 1978]. In particular, the axisymmetric case is investigated. In this limit, our approach permits a uniformly rotating deformed spheroid (post-Newtonian Maclaurin spheroid). These solutions are excluded by the ansatz taken by Chandrasekhar and Elbert, [1974, 1978].

Introduction

Whether or not stationary, non-axisymmetric solutions exist in General Relativity is still an open problem. However, in Newtonian gravity such solutions are known, like the tri-axial Dedekind ellipsoids. Such perfect fluid solutions are stationary due to internal motion where every particle moves on ellipses perpendicular to the shortest semi-axis. The velocity field is given by

\[ \left( v^a \right) = \sqrt{2\pi\mu B_{12}} \left( \frac{a_1}{a_2} x_2, \frac{a_2}{a_1} x_1, 0 \right), \]  

(1)

where \( \mu \) denotes the constant mass density and the index symbols \( A_{ijk...} \) and \( B_{ijk...} \) are discussed at length in § 21 of [Chandrasekhar, 1987]. The one parameter family of the Newtonian Dedekind ellipsoids is characterized by the semi-major axes \( a_1 \geq a_2 \geq a_3 \). Where one of these, e.g. \( a_1 \), is just a scaling freedom and \( a_2/a_1 \) serves as a parameterization. The third axes is obtained using the equation

\[ a_1^2 a_2^2 A_{12} = a_3^2 A_3. \]  

(2)

At the bifurcation point \( a_1 = a_2 \) and respectively \( a_3/a_1 \approx 0.5827241661 \) the Dedekind ellipsoids coincide with the rigidly rotating Maclaurin spheroids. Furthermore, the one parameter family of the non-stationary and tri-axial Jacobi ellipsoids branches off at this point, too.

Constructing a solution similar to the Dedekind ellipsoids in full General Relativity seems not feasible up to now. Nevertheless, the Newtonian Dedekind ellipsoids serve as a natural starting point for answering the question of their existence in a post-Newtonian (PN) approximation scheme. In the 60s and 70s, several authors calculated the first PN orders of the aforementioned Newtonian ellipsoidal figures of equilibrium. For the present work, a series of papers by Chandrasekhar and collaborators is particularly important. In [Chandrasekhar, 1967a] he discussed the first order of the PN Maclaurin ellipsoids and in [Chandrasekhar, 1967b] the first order of the PN Jacobi ellipsoids. In this paper, the relation of these two post-Newtonian figures at the (Newtonian) point of bifurcation was investigated and it was shown, that the axisymmetric limit of the PN Jacobi ellipsoids coincides with a certain PN Maclaurin ellipsoid just as their Newtonian counterparts. This is related to the fact that the PN figures were chosen to rotate uniformly. On the other hand, the PN velocity field of the PN Dedekind ellipsoids chosen in [Chandrasekhar and Elbert, 1978] excludes the possibility of uniform rotation in the axisymmetric limit although it is possible in the axisymmetric case. This restriction seems neither natural...
nor advisable in the context of trying to settle the question as to the existence of relativistic, non-axisymmetric, stationary solutions. The naïve expectation is that the axisymmetric PN Dedekind ellipsoids contain the PN Maclaurin spheroids in the axisymmetric limit (up to arbitrary order). In order to show, that this is possible (at least to first order) we will introduce a more general ansatz for the velocity field than in [Chandrasekhar and Elbert, 1978] and consider its axisymmetric limit. Requiring rigid rotation in the limit, it is shown that a PN Maclaurin spheroid results. To distinguish which formulae hold in the axisymmetric case from those in the latter. In general, the equations and their solutions are different in these two cases. Note that $a_3$ is always understood as a function of $a_1$ and $a_2$ determined by equation (2).

The Axisymmetric Solution of a Generalization to Chandrasekhar and Elbert’s Paper

We consider a generalization of the post-Newtonian Dedekind ellipsoids presented in [Chandrasekhar and Elbert, 1978] (referred to from here on in as Paper I) in which we add post-Newtonian terms to the velocity. We comply with the notation used in Paper I and refer the reader to the definitions there for the various quantities. The post-Newtonian contributions to the velocity, which we introduce here, are

$$\delta v_1 = a_1^2 w_1 x_2 + (q_1 + q) x_1^2 x_2 + r_1 x_2^3 + t_1 x_2 x_3^2,$$

$$\delta v_2 = a_2^2 w_2 x_1 + (q_2 - q) x_1 x_2^2 + r_2 x_1^3 + t_2 x_1 x_3^2,$$

$$\delta v_3 = q_3 x_1 x_2 x_3,$$

where the terms with $w_1$ and $w_2$ have been added for reasons that will be made clear when we discuss the solution. Note that we could eliminate one constant by introducing variables to denote $q_1 + q$ and $q_2 - q$, but choose instead to retain the notation in Paper I.  

Let us in this section consider the axisymmetric case $a_2 = a_1$. That this leads indeed to an axisymmetric solution like in Newtonian gravity will be verified shortly. Then, the index ‘2’ in the index symbols $A_{ijk...}$ and $B_{ijk...}$ can be replaced by ‘1’ as is evident from their definitions [Chandrasekhar, 1987]. Using the relations given in the same book, it is possible to reduce all the index symbols to $A_1$ and $A_2$. At the point $a_2 = a_1$, the value for $A_1$ (and thus $A_2$) is given by (36) in § 17 of [Chandrasekhar, 1987]. Furthermore, equation (2) from [Chandrasekhar and Elbert, 1974] shows us that

$$Q_2 \overset{a}{=} -Q_1.$$  

With these identities the calculations in Paper I can be repeated with the ansatz for the velocity field (3). We shall refer to equation numbers of Paper I by adding a prime. Of course, it is understood that changes due to the generalized ansatz of (3) must be taken into account. We only give the results here and refer the reader for details to [Gürlebeck and Petroff, 2010]. The conclusions drawn from the integrability condition for the pressure and the continuity (cf. (24’) and (38’)) lead to

$$q_3 \overset{a}{=} 0, \quad q_2 \overset{a}{=} -q_1.$$  

Equation (28’) is identically fulfilled in the limit, meaning that $q_1$ is left undetermined. Furthermore, we obtain from (32’) and (37’)

$$r_2 \overset{a}{=} -r_1, \quad t_2 \overset{a}{=} t_1.$$  

\[\text{The three-velocity } v^a \text{ in Paper I does not refer to the spatial components of the four-velocity } u^a = dx^a/d\tau, \text{ but is instead defined as } v^a = dx^a/dt = u^a/u^\tau.\]
Next we turn our attention to the equations implied by requiring that the normal component of the new velocity vanishes on the surface, equations (52')–(56'), and that the pressure vanishes at the surface, (58')\(^2\). Using the results (4)–(6), we can subtract equation (52')-(56') and arrive at

\[ q + q_1 - r_1 \overset{a}{=} \frac{4}{3} Q_1 (4S_3 + S_4). \]  

(7)

The first of the equations (58') becomes in the case \(a_1 = a_2\)

\[ S_1 \overset{a}{=} -4S_3, \]  

(8)

where we made use of (7) from the current paper. With this result, (7) and equation (53') from Paper I\(^3\) we get

\[ q + q_1 - r_1 \overset{a}{=} 0, \quad S_1 - S_2 \overset{a}{=} -\frac{5}{3} S_3. \]  

(9)

The third minus the second of equations (58') is the analogue of equation (101) in [Chandrasekhar, 1967b], whose solution is given by

\[ S_5 \overset{a}{=} -\frac{17a_1^2}{3a_3^2} S_3. \]  

(10)

Furthermore, we can use (56') together with (6) and (8)–(10) to conclude that

\[ t_1 \overset{a}{=} 0, \quad t_2 \overset{a}{=} 0. \]  

(11)

Equation (47') of Paper I tells us that the bounding surface is axisymmetric to the first PN order if and only if (8), the second equation of (9) and (10) hold. To obtain additionally an axisymmetric PN velocity field in the case \(a_1 = a_2\), we also have to require

\[ w_2 \overset{a}{=} -w_1. \]  

(12)

Using what has been shown above, the third equation of (58') in Paper I is used to find the value of \(S_3\). Introducing the eccentricity

\[ e = \sqrt{1 - \frac{a_1^2}{a_2^2}}, \]  

(13)

we define

\[ C := 104e^6 - 444e^4 + 630e^2 - 245, \quad Q_1 \overset{a}{=} -\sqrt{\frac{8e^2(1 - e^2)}{3 + 8e^2 - 8e^4}}. \]  

(14)

With these quantities, one finds an explicit expression for \(S_3\):

\[ S_3 \overset{a}{=} \frac{36e^4}{65C} \left[ \frac{(272e^4 - 244e^2 + 35)Q_1}{8e^2} - \frac{3e^2}{Q_1} r_1 \right]. \]  

(15)

Analogously, we use the fifth of equations (58') to determine \(S_1\):

\[ S_1 \overset{a}{=} \frac{e}{2e^2 - 1} \left[ -\frac{(2864e^8 - 10128e^6 + 14712e^4 - 8120e^2 + 1365)Q_1^2}{26eC} + \frac{e}{3Q_1} w_1 \right] + \frac{4e(224e^6 - 840e^4 + 1170e^2 - 455)}{39CQ_1} r_1. \]  

(16)

The fourth equation of (58') is then identically fulfilled. Our solution at the point \(a_2 = a_1\) has two remaining constants, \(w_1\) and \(r_1\) (although \(q\) and \(q_1\) are not determined, they always appear in the combination \(q + q_1\), which is equal to \(r_1\), cf. (9)).

\(^2\)Please note that we have been unable to reproduce the values from Table 1 in Paper I that result from solving (58'). A detailed discussion can be found in [Gürlebeck and Petroff, 2010].

\(^3\)In (53') of Paper I, the factor \(Q_1\) is missing from the term with \((S_1 - S_2)\).
The Axisymmetric Limit of a Generalization to Chandrasekhar and Elbert’s Paper

Up to now, the equations at the point $a_1 = a_2$ were considered. However, to shed light on the connection of the PN Dedekind ellipsoids and the PN Maclaurin spheroids it is also necessary to investigate the limit $a_2 \to a_1$. The equations listed above are also obtained as limiting relations. But there are two new equations obtained, one of which allows us to determine $\lim_{a_2 \to a_1} q_1$ and the other, say $\lim_{a_2 \to a_1} r_1$.

Equations (24'), (28') and (38') of Paper I provide a system of three linear equations for the quantities $q_1, q_2$ and $q_3$. After solving this linear system for an arbitrary $a_2 < a_1$, the limit $a_2 \to a_1$ can be taken to give

$$q_1 \to -6\sqrt{2}B_{111} \left( 4a_1^2 B_{111} + \frac{a_1^4}{a_3^3} B_{113} \right).$$  \hspace{1cm} (17)

The fourth equation of (58') of Paper I is identically fulfilled for $a_1 = a_2$. Expanding this equation in the quantity $1 - a_2^2/a_1^2$ and demanding that it is satisfied to first order yields

$$\lim_{a_2 \to a_1} r_1 = -\frac{Q_1^3(24e^4 - 12e^2 - 1)}{8e^2(2e^2 + 1)} - \frac{7}{4}\lim_{a_2 \to a_1} w_1.$$  \hspace{1cm} (18)

Note, that we were able to solve the aforementioned equations of Paper I for all $a_2 < a_1$ using our ansatz (3) (cf. [Gürlebeck and Petroff, 2010]. By this means, we showed that the fourth equation of (58') is indeed satisfied to all orders.

Discussion

The solution generated in the axisymmetric case $a_1 = a_2$ depends on two parameters $r_1, w_1$ and in the limit $a_1 \to a_2$ just one $w_1$. In general, these solutions are not uniformly rotating and therefore do not coincide with the PN Maclaurin spheroids. In the limit, uniform rotation is obtained by requiring the velocity field to be shear-free which tells us that

$$r_1 \to 0$$  \hspace{1cm} (19)

must hold. Under this assumption $\lim_{a_2 \to a_1} w_1$ can be read off equation (18).

We now show that with this additional constraint, the solution is indeed equivalent to a certain post-Newtonian Maclaurin ellipsoid which means that the condition of a shear free velocity field is not just necessary but also sufficient. We first explicate what precisely must be shown. There are two degrees of freedom, which amount to the mapping between a Newtonian and post-Newtonian solution and is a matter of convention. For example, one can write the coordinate volume of the star to be

$$V = V_0 + V_1 \delta + \ldots,$$  \hspace{1cm} (20)

where $\delta$ is some relativistic parameter, and then choose to have the PN contribution vanish, $V_1 = 0$. This is the choice that was made in Paper I, [Chandrasekhar, 1967b] and also in Chandrasekhar’s original paper on the post-Newtonian Maclaurin spheroids [Chandrasekhar, 1967a], and which we shall refer to as Paper II. We have followed this convention in the current paper, making it easy to compare our results to those of Paper II. The second freedom one has, was left unspecified in much of Paper II, though Table I lists values with the choice $S_1^M = S_3^M = 0.4$.

\hspace{1cm} *Where necessary, we distinguish the constants of Paper II from those used here by adding the superscript ‘M’."
To compare the surface defined in Paper II with the ansatz made in equation (47') of Paper I using (8)–(10) and (15) we introduce the new coordinate

$$\varpi^2 := x_1^2 + x_2^2.$$  

Then, the surface is defined by

$$0 = \frac{\varpi^2}{a_1^2} + x_3^2 - 1 - \frac{2\pi G \rho a_1^2}{c^2} \left\{ S_1 \left( \frac{\varpi^2}{a_1^2} - 2x_3^2 \right) + S_3 \left[ \frac{5}{3} \left( \frac{\varpi^2}{a_1^2} - x_3^2 \right) - \frac{4}{3} \varpi^4 + \frac{4}{3} \varpi^2 x_3^2 \right] \right\},$$

(22)

The equation for the Newtonian surface $\varpi^2/a_1^2 = 1 - x_3^2/a_3^2$ can be inserted into the PN term above. Comparing the resulting equation with (42) of Paper II shows that these surfaces are identical if

$$S_3 = \frac{9}{13} S_2^M + \frac{3a_3^2}{13a_1^2} S_3^M \quad \text{and}$$

$$S_1 = S_1^M + \frac{16}{13} S_2^M - \frac{a_3^2}{13a_1^2} S_3^M$$

(23)

(24)

hold. As mentioned in that paper, $S_3^M = 0$ may be chosen without loss of generality. Then, equation (22) is satisfied if the values for $S_3^M$ from (99) from Paper II and for $S_3$ from (15) are inserted. The constant $S_1^M$ can be chosen arbitrarily just as with $S_1$ (which depends on $w_1$). If one considers the limit $a_2 \to a_1$ and simultaneously requires that the star rotate uniformly, then (28) provides a unique value for $w_1$, which is equivalent to making a choice for $S_1^M$ different from the one made in Paper II, but no more and no less physically significant.

The most important result of the analysis of the axisymmetric limit is that (18) shows us that the Maclaurin limit ($r_1 = 0$) and the original choice of velocity field in Paper I ($w_1 = w_2 = 0$) are incompatible. While it is possible with that velocity field to find the post-Newtonian Maclaurin solution at the bifurcation point, this solution is not continuously connected to any other solution. When considering the question of the existence or non-existence of non-axially symmetric but stationary solutions, it seems important to retain the possibility of studying a neighbourhood of the axially symmetric and uniformly rotating limit, especially since such solutions are known to exist\(^5\). This possibility was excluded by the approach taken in Paper I.

In a follow-up paper, we intend to tackle the problem with a more general approach that lends itself better to proceeding to higher post-Newtonian orders, is not as restrictive in the solutions it permits and allows one to show that the singularity discussed in Paper I is an artefact of the specific method chosen and not an inherent property of the post-Newtonian Dedekind solutions (cf. [Gürlebeck and Petroff, 2010]).

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\(^5\)As far as we know, there exists no formal proof demonstrating the existence of such solutions. Steps in that direction were taken by [Heilig, 1995] and the existence has been demonstrated by many groups that are able to solve Einsteins equations numerically to extremely high accuracy, see e.g. [Ansorg, Kleinwächter and Meinel, 2003].
GÜRLEBECK AND PETROFF: AXISYMMETRIC LIMIT OF PN-DEDEKIND ELLIPSOIDS

References


