Extremal Energy Shifts from a Radiating Ring
Near a Rotating Black Hole

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Abstract. We present a semi-analytical solution to the problem of finding minimal and maximal energy shifts of radiation originating from an accretion ring near a black hole. The solution is given implicitly in terms of a set of two coupled equations for two unknowns – the constants of photon motion. These are functions of three parameters (the spin of black hole, the emission radius, and the inclination of an observer), which we solve numerically to obtain an explicit form of the extremal energy shifts. Results are presented also graphically via contour graphs.

Introduction

Emission from inner regions of accretion disks around black holes provides information about matter in extreme conditions. Kα spectral line of iron, broadened and skewed by fast orbital motion and redshifted by strong gravitational field, has been used to constrain parameters of the black hole, both in active galactic nuclei [Fabian et al., 2000; Reynolds & Nowak, 2003; Miller, 2007] and Galactic X-ray binaries containing black-holes [Miller et al., 2002; McClintock & Remillard, 2006].

It is most relevant to know the expected energy range of iron line profiles, depending on the basic model parameters – i.e., radius where the emission takes place, inclination angle of the observer, and angular momentum of the black hole. So our goal was to find a formula, where the output parameters would be minimum and maximum values of the change of the energy of the radiation from the surroundings of black hole. Knowing such a formula could help us to constraint parameters of accreting black holes.

Kerr metric

The gravitational field of a stationary, axisymmetric and rotating black hole is described by Kerr metric [Chandrasekhar, 2004; Kerr, 1963]. The form of Kerr metric in Boyer-Lindquist co-ordinates \((t, r, \theta, \phi)\) and geometrised units \((c = G = M = 1)\) is

\[
ds^2 = -(1 - \frac{2r}{\Sigma}) dt^2 - \frac{4ar}{\Sigma} \sin^2 \theta dtd\phi + \frac{A}{\Sigma} \sin^2 \theta d\phi^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2,
\]

where

\[
\Sigma = r^2 + a^2 \cos^2 \theta; \Delta = r^2 - 2r + a^2; A = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta.
\]

The metric depends only on one parameter, angular momentum of the black hole \(a\).

The minimum allowed radius of a stable circular equatorial orbit, so called marginally stable orbit [Bardeen et al., 1972], is given by the roots of the equation

\[
r^2 - 6r \mp 8a\sqrt{r} - 3a^2 = 0.
\]

The roots are

\[
r_{ms} = 3 + Z_2 \mp [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2},
\]

where \(Z_1 = 1 + (1 - a^2)^{1/3}[(1 + a)^{1/3} + (1 - a)^{1/3}]; Z_2 = (3a^2 + Z_1^2)^{1/2},\) where the upper sign refers to co-rotating and the lower to counter-rotating orbits.
Photon propagation in Kerr metric

The path of photons (null geodesic) in Kerr metric is completely described by three constants of motion: the total energy $E$, the azimuthal angular momentum $L_A$, and Carter’s constant $Q$. We can further reduce the number of constants by re-normalizing $L_A$ and $Q$ with respect to energy $E$, $\lambda = \frac{L_A}{E}$, $q^2 = \frac{Q}{E^2}$.

Further, a null geodesic must satisfy the Carter equation [Carter, 1968]

$$\pm \int_r \frac{dr}{\sqrt{R(r, \lambda, q^2)}} = \pm \int_{\mu} \frac{d\mu}{\sqrt{\Theta(\mu, \lambda, q^2)}}$$  \hspace{1cm} \text{(5)}

where

$$R(r) = r^4 + (a^2 - \lambda^2 - q^2)r^2 + 2[q^2 + (\lambda - a)^2]r - a^2q^2$$  \hspace{1cm} \text{(6)}

and

$$\Theta(\mu, \lambda, q^2) = q^2 + (a^2 - \lambda^2 - q^2)\mu^2 - a^2\mu^4,$$  \hspace{1cm} \text{(7)}

where we suppose the substitution $\mu = \cos \theta$. The left side of the eq. (5) describes the motion in radial direction and the right side the motion in latitudinal direction.

The roots of $R(r)$ and $\Theta(\mu)$ correspond to the turning points in radial and latitudinal directions respectively [Čadež et al., 1998]. The polynomial $R(r)$ can be expressed in the form $R = (r - r_1)(r - r_2)(r - r_3)(r - r_4)$, where

$$r_1 = \frac{1}{2}F + \frac{1}{2}\sqrt{D_-}; \quad r_2 = \frac{1}{2}F - \frac{1}{2}\sqrt{D_-}; \quad r_3 = -\frac{1}{2}F + \frac{1}{2}\sqrt{D_+}; \quad r_4 = -\frac{1}{2}F - \frac{1}{2}\sqrt{D_+}$$  \hspace{1cm} \text{(8)}

are the roots of $R(r) = 0$. The polynomial $\Theta(\mu)$ can be expressed in the form $\Theta(\mu) = a^2(\mu_+^2 + \mu_-^2)(\mu_+^2 - \mu_-^2), \ q^2 > 0,$ where

$$\mu_\pm^2 = \frac{1}{2a^2}\{[(\lambda^2 + q^2 - a^2)^2 + 4a^2q^2]^{1/2} \mp (\lambda^2 + q^2 - a^2)\}. \hspace{1cm} \text{(9)}$$

The expressions for $r_1, r_2, r_3, r_4$ are

$$A = (a^2 - \lambda^2 - q^2); \quad B = (a - \lambda)^2 + q^2; \quad C = A^2 - 12a^2q^2; \quad D = 2A^3 + 72a^2q^2A + 108B^2, \hspace{1cm} \text{(10)}$$

$$E = \frac{1}{3}\left[\left(D - \sqrt{-4C^3 + D^2}\right)^{\frac{1}{2}} + \left(D + \sqrt{-4C^3 + D^2}\right)^{\frac{1}{2}}\right]; \quad F = \sqrt{-\frac{2}{3}A + E} \hspace{1cm} \text{(11)}$$

and

$$D_\pm = -\frac{4}{3}A - E \pm \frac{4B}{F}. \hspace{1cm} \text{(12)}$$

The roots $r_1$ and $r_2$ of the polynomial $R(r)$ can be real or complex, whereas the roots $r_3$ and $r_4$ are always real.

The solution of Carter equation

The integrals in Carter equation (5) can be expressed in the form of the elliptical integrals of first kind [Abramowitz et al., 1965; Byrd et al., 1971]

$$F(\varphi, k) = \int_{\varphi}^{\varphi} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}. \hspace{1cm} \text{(13)}$$

The explicit form of the integral depends on whether the roots $r_1$ and $r_2$ are real or complex and whether the photon passes through the turning points or not.

In our problem we consider direct rays going from the equatorial ring (with the radius $r_e$) toward a distant observer at infinity with the inclination $\theta_o$. Only certain combinations of roots are therefore relevant, and we do not have to discuss all possibilities.
The radial integral, real roots

The case of real roots is

\[ \int_{r_e}^{\infty} \frac{dr}{\sqrt{R(r, \lambda, q^2)}} = g_r[F(\varphi_o, k_r) \pm F(\varphi_e, k_r)], \]

(14)

where

\[ g_r(\lambda, q^2) = \frac{2}{\sqrt{(r_1 - r_3)(r_2 - r_4)}}; \quad k_r(\lambda, q^2) = \frac{(r_2 - r_3)(r_1 - r_4)}{(r_1 - r_3)(r_2 - r_4)}. \]

(15)

\[ \varphi_o(\lambda, q^2) = \arcsin \left( \frac{r_2 - r_4}{r_1 - r_4} \right); \quad \varphi_e(\lambda, q^2) = \arcsin \left[ \sqrt{(r_2 - r_4)(r_e - r_1)} \right] \left[ \frac{(r_1 - r_4)(r_e - r_2)}{(r_1 - r_4)(r_e - r_2)} \right]. \]

(16)

The upper sign refers to the case of a photon that passes through the turning point and the lower sign refers to the case of a photon that does not pass through the turning point in the radial direction.

The radial integral, complex roots

We suppose \( r_1, r_2 \) are complex and \( r_3, r_4 \) are real, then the roots \( r_1, r_2 \) can be written in the form \( r_1 = u + iv \), \( r_2 = u - iv \), where \( u = \frac{1}{2} F \), \( v = \frac{1}{2} \sqrt{T} \). The expression for this case is

\[ \int_{r_e}^{\infty} \frac{dr}{\sqrt{R(r, \lambda, q^2)}} = g_r[F(\varphi_o, k_r) - F(\varphi_e, k_r)], \]

(17)

where

\[ g_r(\lambda, q^2) = \frac{1}{\sqrt{AB}}; \quad k_r(\lambda, q^2) = \frac{(A + B)^2 - (r_3 - r_4)^2}{4AB}, \]

(18)

\[ \varphi_o(\lambda, q^2) = \arccos \left( \frac{A - B}{A + B} \right); \quad \varphi_e(\lambda, q^2) = \arccos \left[ \frac{(A - B)r_e + r_3B - r_4A}{(A + B)r_e - r_3B - r_4A} \right], \]

(19)

\[ A(\lambda, q^2) = [(r_3 - u)^2 + v^2]^{1/2}; \quad B(\lambda, q^2) = [(r_4 - u)^2 + v^2]^{1/2}. \]

(20)

The latitudinal integral

The form of the latitudinal integral is

\[ \int_{0}^{\mu_e} \frac{d\mu}{\sqrt{\Theta(\mu, \lambda, q^2)}} = \frac{g_\mu}{a} F(\psi, k_\mu) \]

(21)

if the photon does not pass through the turning point. For the case with transit through the turning point it is

\[ \int_{0}^{\mu_e} \frac{d\mu}{\sqrt{\Theta(\mu, \lambda, q^2)}} = \frac{g_\mu}{a} [2K(k_\mu) - F(\psi, k_\mu)], \]

(22)

where

\[ g_\mu(\lambda, q^2) = \frac{1}{\sqrt{\mu_+^2 + \mu_-^2}}; \quad k_\mu(\lambda, q^2) = \frac{\mu_+^2}{\mu_+^2 + \mu_-^2}; \quad \psi(\lambda, q^2) = \arcsin \left[ \sqrt{\frac{\mu_+^2(\mu_+^2 + \mu_-^2)}{\mu_+^2(\mu_+^2 + \mu_-^2)}} \right] \]

(23)

and \( K(k_\mu) = F(\frac{\pi}{2}, k_\mu) \).
The minimum and maximum of energy shift

The energy shift can be calculated as the ratio \( g \) between the observed \( E_o \) and emitting \( E_e \) energy

\[
g = \frac{E_o}{E_e}. \tag{24}
\]

If we suppose an emitting particle in Kerr metric with the four velocity \( u = u^I(1, 0, 0, \Omega) \), where \( u^I = \left[ 1 - \frac{2\mu}{c^2} (1 - \alpha \Omega \sin^2 \theta_c) - (\Gamma^2_c + \alpha^2) \Omega^2 \sin^2 \theta_c \right]^{-1/2} \) and \( \Omega = \frac{1}{\sqrt{\Gamma^2_c + \alpha}} \) (Keplerian angular velocity), then the ratio \( g \) is

\[
g = \frac{1}{\alpha^2} \frac{1}{1 - \lambda \Omega}. \tag{25}
\]

The extreme values of \( g \) are required to meet the conditions of the Carter equation (5). We used the Lagrange multipliers \( \alpha \) to find their values. Define the Lagrangian as

\[
\Lambda(\lambda, q^2, \alpha) = \frac{1}{u^I} \frac{1}{1 - \lambda \Omega} - \alpha \int_{r_e}^{\infty} \frac{dr}{\sqrt{R(r, \lambda, q^2)}} + \alpha \int_0^{\mu_o} \frac{d\mu}{\sqrt{(\mu, \lambda, q^2)}}. \tag{26}
\]

The partial derivatives of the Lagrangian must satisfy

\[
\frac{\partial}{\partial \lambda} \Lambda(\lambda, q^2, \alpha) = 0; \quad \frac{\partial}{\partial q^2} \Lambda(\lambda, q^2, \alpha) = 0; \quad \frac{\partial}{\partial \alpha} \Lambda(\lambda, q^2, \alpha) = 0. \tag{27}
\]

If we suppose \( \alpha \neq 0 \) we get from the three equations (27) for three unknowns \( \lambda, q^2, \alpha \) two equations

\[
f_1 = \int_{r_e}^{\infty} \frac{dr}{\sqrt{R(r, \lambda, q^2)}} - \int_0^{\mu_o} \frac{d\mu}{\sqrt{(\mu, \lambda, q^2)}} = 0, \tag{28}
\]

\[
f_2 = \frac{\partial}{\partial q^2} \left[ \int_{r_e}^{\infty} \frac{dr}{\sqrt{R(r, \lambda, q^2)}} - \int_0^{\mu_o} \frac{d\mu}{\sqrt{(\mu, \lambda, q^2)}} \right] = 0 \tag{29}
\]

for two unknowns \( \lambda, q^2 \), because the equations \( f_1, f_2 \) do not contain \( \alpha \) and that’s why we do not need the derivative \( \frac{\partial}{\partial \lambda} \Lambda(\lambda, q^2, \alpha) \). The found value of \( \lambda \) will correspond to the extreme value of the function \( g \) (25).

However, the set of the eqs. (28), (29) is nonlinear and we need a numerical method to solve it. We used the Newton-Raphson method, which is written in terms of Taylor expansion about the root

\[
f_1(\lambda, q^2) = 0 = f_1(\lambda_n, q_n^2) + (\lambda - \lambda_n) \frac{\partial f_1}{\partial \lambda}(\lambda_n, q_n^2) + (q^2 - q_n^2) \frac{\partial f_1}{\partial q^2}(\lambda_n, q_n^2) + O((\lambda - \lambda_n)^2 + (q^2 - q_n^2)^2),
\]

\[
f_2(\lambda, q^2) = 0 = f_2(\lambda_n, q_n^2) + (\lambda - \lambda_n) \frac{\partial f_2}{\partial \lambda}(\lambda_n, q_n^2) + (q^2 - q_n^2) \frac{\partial f_2}{\partial q^2}(\lambda_n, q_n^2) + O((\lambda - \lambda_n)^2 + (q^2 - q_n^2)^2). \tag{30}
\]

Define \( \Delta \lambda_n = \lambda - \lambda_n \) and \( \Delta q_n^2 = q^2 - q_n^2 \), then

\[
\Delta \lambda_n \frac{\partial f_1}{\partial \lambda}(\lambda_n, q_n^2) + \Delta q_n^2 \frac{\partial f_1}{\partial q^2}(\lambda_n, q_n^2) \approx -f_1(\lambda_n, q_n^2), \tag{32}
\]

\[
\Delta \lambda_n \frac{\partial f_2}{\partial \lambda}(\lambda_n, q_n^2) + \Delta q_n^2 \frac{\partial f_2}{\partial q^2}(\lambda_n, q_n^2) \approx -f_2(\lambda_n, q_n^2). \tag{33}
\]

These equations are linear for \( \Delta \lambda_n, \Delta q_n^2 \), which we get from one iteration and the values \( \lambda_{n+1} = \lambda_n + \Delta \lambda_n, q_{n+1}^2 = q_n^2 + \Delta q_n^2 \) can be used as seeds for next iteration.
Figure 1. The contour graph showing the curves of constant radius of emission \( r_{em} \) (thick solid line), and of constant spin \( a \) of the black hole (thin). The corresponding values are given with each line. The dashed line represents the marginally stable orbit \( r_{ms} \). The minimum energy shift \( g_{min} \) is given on the abscissa, while the ordinate gives the maximum energy shift \( g_{max} \). The observer view angle is \( \cos = 0.07 \) (\( i \sim 86 \) deg).

Figure 2. The contour graph as in previous figure but for \( \cos = 0.40 \) (\( i \sim 66 \) deg).
Figs. 1, 2 show the computed extremes of the energy shifts for different positions of the observer. The contours are close to each other for lower inclination and cannot be distinguished from each other, because the dependence of maximum values on spin \( a \) is very small. These graphs help us to connect the values of emission radius and the black hole spin with the extremal shifts of observed energy. In other words, knowing \( g_{\text{max}} \) and \( g_{\text{min}} \) allows us to infer immediately the radius of the emission \( r_{\text{em}} \) ring and the spin \( a \).

**Discussion**

The minimum value of energy shift \( g_{\text{min}} \) goes to zero for larger spin \( a \), if we suppose emitting radius the marginally stable orbit \( r_{\text{ms}} \). The dependence on the inclination is very small. The range of minimum values is almost same for all inclinations (see Figs. 1, 2). The maximum value of energy shift \( g_{\text{max}} \) mainly depends on the inclination of observer and the dependence on spin \( a \) is small. That’s why the contours are close each other for lower inclinations. The origin of energy shift is gravitational redshift and Doppler shift for large inclinations. For small inclinations it is only gravitational redshift, hence the values are smaller than one. The computed values are consistent with the results of the work [Dovčiak et al., 2004].

**Conclusion**

The extremes must satisfy the Carter equation (5) that is represented in the form of the elliptical integrals, eqs. (14), (17), (21). The best way to find these extremes is to use the procedure of Lagrange multipliers. This procedure gave us two nonlinear equations (28), (29) for two unknowns: the constants of photon motion, \( \lambda \) and \( q^2 \). This set of equations can be solved only by the help of numerical methods, e.g. Newton-Raphson method. The resulting extremes of energy shifts are presented via contour graphs (Figs. 1, 2) for different view angles of the observer.

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**References**