The Isodynamic Points of the Tetrahedron

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Abstract. The isodynamic points of the triangle, the Lemoine axis and the Brocard axis are well-known properties of the triangle. In the contribution, we will describe spatial analogies of the mentioned terms – the isodynamic points of the tetrahedron, the Lemoine plane and the Brocard axis of the tetrahedron. Further, we will provide synthetic proofs of described properties of tetrahedron and their spatial generalizations. Moreover, we will show not only similarities between properties of the triangle and the tetrahedron but also analogies between proving methods in the plane and in the space.

Introduction

The tetrahedron is a natural spatial analogy to the triangle. In contrast to the properties of the triangle, some of the properties of the tetrahedron are not widely known. Various spatial analogies of the properties of the triangle and properties of the tetrahedron are described in many articles and short works (e.g. [Coolidge] or [Havlicek]). However, results in this field of geometry have not been thoroughly summarized so far. Proofs and corresponding illuminating figures are missing in many articles and publications. Computational approach is preferred by the majority of authors, but many calculations and the lack of figures make it harder to see the correspondence between planar and spatial geometry.

Spatial imagination is needed when describing properties of a tetrahedron. Deduction of classical synthetic proofs is not so easy as in the planar case. It is possible to use modern modeling and mathematical software to create true images for illustration of a spatial problem. A text with such images is more "readable" even for readers without the deep knowledge of spatial geometry.

The main aim of this contribution is to describe a few interesting properties of the tetrahedron – the spatial analogies of the properties of the triangle. Moreover we will provide synthetic proofs of these properties, which were omitted in the majority of accessible sources. The correspondence between proofs in the spatial and planar case is emphasized whenever possible.

The paper contains full-color figures, which should be viewed in electronic version of the paper (accessible on web od "WDS 2010" conference).

The isodynamic points, the Lemoine axis, the Brocard axis of the triangle

At first, let us recall some notation in triangle $ABC$. Lines $o_a$, $o_b$, $o_c$ are the bisectors of the interior angles at vertices $A$, $B$, $C$. Lines $o_a'$, $o_b'$, $o_c'$ denote the bisectors of the exterior angles of the triangle at vertices $A$, $B$, $C$. Points $D_a$ and $D_a'$ are the points of intersection of $o_a$ and $o_a'$ with line $BC$. Points $D_b$, $D_b'$, $D_c$ and $D_c'$ are defined in a similar fashion – see Fig. 1.

The Apollonian circles of triangle $ABC$ are circle $A_a$ with diameter $D_aD_a'$, circle $A_b$ with diameter $D_bD_b'$ and circle $A_c$ with diameter $D_cD_c'$ – see Fig. 1.

The isodynamic points of triangle $ABC$ are the common points of intersection of the three Apollonian circles of triangle $ABC$. Let us denote these point $I$ and $I'$.

The centers of the Apollonian circles, i.e., the midpoints of line segments $D_aD_a'$, $D_bD_b'$, $D_cD_c'$, are collinear. These points lie on the Lemoine axis of triangle $ABC$ – see line $ℓ$ in Fig. 1.

The Brocard axis $r = II'$ is perpendicular to the Lemoine axis $ℓ$. The center $S$ of the circle $k$ circumscribed to the triangle $ABC$ is point of the Brocard axis.

Spatial generalizations

A spatial generalization of the angle bisector is the plane of symmetry of the faces of a tetrahedron. There are six pairs of planes of symmetry in a tetrahedron. In Fig. 2, we see planes $ω$ and $ω'$ – the planes of symmetry of faces $BCD$ and $ACD$. Points $X$ and $X'$ are the points of intersection of $ω$ and $ω'$ with line $AB$.

A spatial generalization of the Apollonian circle is the Apollonian sphere $A_{AB}$ with diameter $XX'$. There are six Apollonian spheres $A_{AB}$, $A_{BC}$, $A_{AC}$, $A_{AD}$, $A_{BD}$, $A_{CD}$ in a tetrahedron.
All six Apollonian spheres of a tetrahedron have two common points of intersection $I$ and $I'$ – the isodynamic points of a tetrahedron. The centers of spheres $A_{AB}, A_{BC}, A_{AC}, A_{BD}, A_{CD}$ are coplanar – they lie in the Lemoine plane $\lambda$ of a tetrahedron – see Fig. 3. Line $r = II'$ is perpendicular to plane $\lambda$ and passes through the center of the circumsphere of $ABCD$ – $r$ is the Brocard axis of the tetrahedron $ABCD$. 
Figure 3. One of the isodynamic points and the Lemoine plane of a tetrahedron.

Synthetic proof – basic ideas

Apollonian definition of the circle

At first we should explain why circles $A_a$, $A_b$ and $A_c$ are denoted as Apollonian circles. There is a connection between this notation and the so-called Apollonius’ definition of the circle.\(^1\)

Further we recall the property of the bisectors of the triangle angles. For the points of intersection $D_c$ and $D_c'$ of bisectors $o_γ$ and $o_γ'$ with line $AB$ holds following relation – see Fig. 4 b).

\[
\frac{|D_c A|}{|D_c B|} = \frac{|D_c' A|}{|D_c' B|} = \frac{|CA|}{|CB|}
\]

The Apollonian circle $A_c$ is therefore the circle according to Apollonius’ definition. It is obvious that $A_c$ passes through vertex $C$ of triangle $ABC$.

There is natural spatial generalization of the Apollonius’ definition of the circle. Let points $A$ and $B$ in the space be given. The locus of points $P$ such that $\frac{|PA|}{|PB|}$ is constant, is the sphere.

For points $X$ and $X'$ (points of intersection of planes of symmetry $ω$ and $ω'$ with the line $AB$ – see Fig. 2) holds following relation

\[
\frac{|XA|}{|XB|} = \frac{|X'A|}{|X'B|} = \frac{S_β}{S_α},
\]

where $S_β$ and $S_α$ are areas of the triangles $DCA$ and $DCB$.

Therefore the above defined Apollonian sphere of the tetrahedron is the sphere according to generalized Apollonius’ definition.

The existence of the isodynamic points

In any triangle $ABC$, the Apollonian circles $A_a$ and $A_b$ have common points of intersection. Let us denote these points of intersection $I$ and $I'$. From the incidence of $I$ and $I'$ with circles $A_a$ and $A_b$\(^1\)

\(^1\)Let points $A$ and $B$ be given. The locus of points $P$ such that $\frac{|PA|}{|PB|}$ is constant, is the circle – see Fig. 4 a).
SRUBAŘ: THE ISODYNAMIC POINTS OF THE TETRAHEDRON

Figure 4. a) The Apollonius’ definition of the circle. b) The circle $A_c$ is a circle according to Apollonius’ definition.

Figure 5. The circumcircle is perpendicular to the Apollonian circle.

follows

$$\{I, I'\} \in A_a \Rightarrow \frac{|IA|}{|IB|} = \frac{|I'A|}{|IB|} = \frac{|CA|}{|CB|}, \quad \{I, I'\} \in A_b \Rightarrow \frac{|IC|}{|IA|} = \frac{|I'C|}{|IA|} = \frac{|BC|}{|BA|}.$$  

By multiplying this ratios we get

$$\frac{|IA|}{|IB|} \cdot \frac{|IC|}{|IA|} = \frac{|I'A|}{|IB|} \cdot \frac{|I'C|}{|IA|} = \frac{|CA|}{|CB|} \cdot \frac{|BC|}{|BA|} \Rightarrow \frac{|IC|}{|IB|} = \frac{|I'C|}{|IB|} = \frac{|CA|}{|BA|}.$$  

Thus points $I$ and $I'$ are points of the apollonian circle $A_c$, i.e. common points of all three Apollonian circles.

The Lemoine axis, the Brocard axis

From the existence of two common points of three circles $A_a, A_b, A_c$ directly follows the collinearity of their centers, i.e. the existence of the Lemoine axis $\ell$. The line $r = II'$ is so-called radical axis of circles $A_a, A_b, A_c$ and it is obviously perpendicular to the line $\ell$. Line $r$ is the Brocard axis of the triangle $ABC$. For more information about circle power, radical axis and radical center see [Weisstein].

Further we will show that Brocard axis passes through the circumcenter $S$ of triangle $ABC$. At first we need to recall one important property of the Apollonian circle. Each of Apollonian circles $A_a, A_b, A_c$ is orthogonal to the circumcircle $k$ of the triangle $ABC$. For instance the tangent $t$ of the circumcircle $k$ in the vertex $C$ is perpendicular to the tangent $u$ of the Apollonian circle $A_c$ in the vertex $C$ – see Fig. 5.

From the orthogonality of $k$ to all three Apollonian circles follows that center $S$ of $k$ has the same power to all three Apollonian circles. Thus point $S$ lies on the common radical axis of $A_a, A_b$ and $A_c$, i.e. $S$ is point of the Brocard axis $r$ of the triangle $ABC$.

The isodynamic points of the tetrahedron

While proving the existence of the isodynamic points of the tetrahedron we will partly follow the planar proof.
Let us denote $A_X$, $A_Y$, and $A_Z$ the circles of intersection of the apollonian spheres $A_{AB}$, $A_{BC}$ and $A_{AC}$ with the plane $\delta(A, B, C)$, respectively. Further, let us denote $I_5$ and $I'_5$ the common points of the circles $A_X$ and $A_Y$. From the incidence of $I_5$ and $I'_5$ with circles $A_X$ and $A_Y$ (i.e. with spheres $A_{AB}$ and $A_{BC}$) follows

$$\{I_5, I'_5\} \in A_{AB} \Rightarrow \frac{|AI_5|}{|BI_5|} = \frac{|AI'_5|}{|BI'_5|} = \frac{S_\beta}{S_\alpha}, \quad \{I_5, I'_5\} \in A_{BC} \Rightarrow \frac{|BI_5|}{|CI_5|} = \frac{|BI'_5|}{|CI'_5|} = \frac{S_\gamma}{S_\beta}.$$  

By multiplying previous relations we get

$$\frac{|AI_5|}{|BI_5|} \cdot \frac{|CI_5|}{|BI_5|} = \frac{|AI'_5|}{|CI'_5|} \Rightarrow \frac{|AI_5|}{|CI_5|} = \frac{|AI'_5|}{|CI'_5|} = \frac{S_\gamma}{S_\alpha}.$$

Thus points $I_5$ and $I'_5$ are points of the Apollonian sphere $A_{AC}$, i.e. points of the circle $A_Z$. Points $I_5$ and $I'_5$ are two common points of three circles $A_X$, $A_Y$, $A_Z$ – see Fig. 6.

Note: Circles $A_X$, $A_Y$, $A_Z$ are not the Apollonian circles of the triangle $ABC$ and points $I_5$ and $I'_5$ are not the isodynamic points of $ABC$.

Circles $A_X$, $A_Y$, $A_Z$ have collinear centers (line $\ell_5$ is not the Lemoine axis of $ABC$) and share common radical axis $r_5$. These circles are the main circles of the Apollonian spheres $A_{AB}$, $A_{BC}$, $A_{AC}$. Therefore spheres $A_{AB}$, $A_{BC}$, $A_{AC}$ have one common circle $a_5$. There is such circle for Apollonian spheres with centers in one face of the tetrahedron. Thus there are four circles $a_5$, $a_5', a_\beta$, and $a_\gamma$. Moreover there are four lines $\ell_\alpha$, $\ell_\beta$, $\ell_\gamma$ and $\ell_5$, each of them passing through centers of Apollonian spheres in corresponding face of the tetrahedron.

Each two of lines $\ell_\alpha$, $\ell_\beta$, $\ell_\gamma$, $\ell_5$ have one common point and there are not three of these lines with one common point. Therefore lines $\ell_\alpha$, $\ell_\beta$, $\ell_\gamma$ and $\ell_5$ are coplanar – lie in the plane $\lambda$ – the Lemoine plane of tetrahedron see Fig. 7 a).

Each of circles $a_\alpha$, $a_\beta$, $a_\gamma$ and $a_5$ lie in plane perpendicular to both the Lemoine plane and the plane of the face of the tetrahedron. Therefore these circles have two common points $I$ and $I'$ – the isodynamic points of the tetrahedron $ABCD$. The line $r = II'$ is obviously perpendicular to the Lemoine plane $\lambda$, $r$ is the Brocard axis of $ABCD$ – see Fig. 7 b).

Similarly to planar case can be proven that circles $A_X$, $A_Y$, $A_Z$ are perpendicular to circumcircle $k_5$ of the triangle $ABC$. Therefore the center $S_5$ of $k_5$ lie on the common radical axis $r_5$. Point $S_5$ is perpendicular projection of the center $S$ of the circumsphere of $ABCD$ to the plane $\delta(A, B, C)$.

Thus $S$ lie in the plane of the circle $a_5$ and analogically in planes of circles $a_\alpha$, $a_\beta$ and $a_\gamma$. These planes have one common line $r = II'$. Therefore point $S$ has to lie on $r$ – the Brocard axis of $ABCD$.
Figure 7. a) The Lemoine plane, b) The isodynamic points and Brocard axis of the tetrahedron.

Figure 8. Point $S_δ$ lies on the line $r_δ$. The circumcenter of tetrahedron lie in plane of $a_δ$.

Conclusion

The main goal of this contribution is to show that there are not only analogues between the properties of the triangle and the tetrahedron, but also the similarities in proofs in planar and spatial case can be found. These similarities (between both properties and proofs) could be the way how to show the possibility of using methods of planar geometry in space, what we believe could be very helpful for students of geometry with weak spatial imagination. Thus the similarities between the triangle and the tetrahedron could be (at least for some students) the bridge from planar to spatial geometry.

References

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