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Ancient Indian Mathematics

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Abstract. The aim of this article is to present a short outline of early Indian mathematics. I mean to summarize the results and contributions of Indian mathematics which were made in the period from the first civilization in Indian subcontinent to the 5^{th} century AD when classical era of Indian mathematics began.

Indus valley civilization

The first use of mathematics in the Indian subcontinent was in the Indus valley and dates as far back as 3000 BC [*Wheeler*]. The earliest known urban Indian culture was at Harappa in the Punjab and at Mohenjodaro near the Indus River. Excavations at Mohenjodaro, Harappa and the surrounding area of the Indus River discovered evidence of the use of basic mathematics. The mathematics used by the early Harappan civilization had mostly practical intent and was concerned with weights and measuring scales. Excavations present knowledge of basic geometry.

This culture also produced artistic designs. On carvings there is evidence that these people could draw concentric and intersecting circles and triangles. The further using of circles in the Harappan decorative design can be found at the pictures of bullock carts, the wheels of which had perhaps a metallic band wrapped round the rim. It clearly points to the knowledge of the ratio of the length of the circumference of the circle and its diameter, and thus of the value of π.

The Harappans adopted a uniform system of weights and measures [*O'Connor, Robertson*]. Detail analysis discovered that weights corresponding to ratios of 0.05, 0.1, 0.2, 0.5, 1, 2, 5, 10, 20, 50, 100, 200, and 500 were used. The existence of a graduated system of accurately marked weights shows the development of trade and commerce in Harappan society.

Some appliances for the measurement of length were discovered. Also a remarkably accurate decimal ruler known as the Mohenjodaro ruler is very interesting. Its subdivision has a maximum error of just 0.005 inches at a length of 1.32 inches. The length have been named the *Indus inch*. Another scale was discovered when a bronze rod was found which was marked in lengths of 0.367 inches. The accuracy with which these scales are marked is certainly surprising. And 100 units of this measure is 36.7 inches which is the measure of a stride. Measurements of the ruins of buildings which have been excavated show that these units of length were accurately used by the Harappans in construction.

Vedic period

Aryan tribes from the North of Indian subcontinent invaded and destroyed the Harappan culture around 2000 BC. They founded the Vedic religion. And thanks their works we gain the first literary evidence of Indian culture including mathematics. The word Vedic comes from the collections of sacred texts known as Vedas. Mathematics and astronomy first appear in Vedic works during the 2^{nd} millennium BC.

The word *ganita* first appears in Vedic works. The term literally means the science of calculation. It is basically the Indian equivalent of the word mathematics.

The mathematical parts of Vedic works show the surprising development of mathematics. We can find the description of geometric shapes (including triangles, rectangles, squares, trapezia and circles), the solution of the problem of equivalence of area, squaring the circle and

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vice-versa, early forms of the Pythagoras' theorem, estimations for π etc.

Sacrificial rites were the main feature of the Vedic religion. There was a ritual which took place at an altar where food, also sometimes animals, were sacrificed. The Sulbasutras are vedic works which give rules for the construction of fire altars. If the ritual sacrifice should be successful then the altar had to conform to very precise measurements. There were two types of sacrificial rites, one being a large public gathering while the other was a small family affair. Different types of altars were necessary for the two different types of ceremony.

Certainly the Sulbasutras do not contain any proofs of the rules which they describe. Some of the rules, such as the method of constructing a square of area equal to a given rectangle, are exact. Others, such as constructing a square of area equal to a given circle, are approximations.

The most important mathematical manuscripts are the *Baudhayana Sulbasutra* written about 800 BC, the *Apastamba Sulbasutra* written about 600 BC, the *Manava Sulbasutra* written about 750 BC and the *Katyayana Sulbasutra* written about 200 BC.

The Sulbasutras are really construction manuals for geometric shapes such as squares, circles, rectangles, etc. The first result which was clearly known to the authors is the Pythagoras' theorem [*Juškevič*]. The Baudhayana Sulbasutra gives only a special case of the theorem:

> *The rope which is stretched across the diagonal of a square produces an area double the size of the original square.*

The Katyayana Sulbasutra however, gives a more general version:

> *The rope which is stretched along the length of the diagonal of a rectangle produces an area which the vertical and horizontal sides make together.*

The Pythagoras' theorem is used frequently and there are many examples of Pythagorean triples in the Sulbasutras. For example (5, 12, 13), (12, 16, 20), (8, 15, 17), (15, 20, 25), (12, 35, 37), (15, 36, 39).

The construction of the square equal in area to two given unequal squares is also based on the Pythagoras' theorem $((AB)^2 + (AS)^2 = (BS)^2)$.

> *Separate a parallel band of the width of the smaller square from the bigger square. The rope which is stretched diagonally across the band unites both (the squares).*

The problem of making a square whose area is equal to a difference of two given squares is solved similarly. (The Pythagoras' theorem $(SP)^2 - (SA)^2 = (AP)^2$ is used.)

The next construction is to find a square equal in area to a given rectangle.

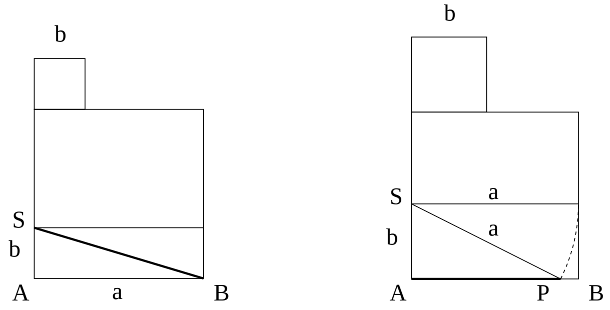

Figure 1. The square equal in area to the sum of two squares (left), the square equal in area to the difference of two squares (right).

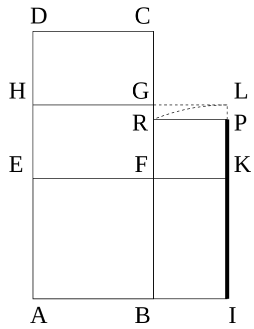

Figure 2. The square equal in area to the given rectangle.

The rectangle ABCD is given. Let E be marked on AD so that AE = AB. Then complete the square ABFE. Now bisect ED at H and divide the rectangle EFCD into two equal rectangles with the line HG. Now move the rectangle HGCD to the position FBIK. Complete the square AILH. The required square is equal in area to the difference of the squares AILH and FKLG. Now rotate IL about I so that it touches BG at R, then IL = IR. Now draw RP parallel to GL such that P is on IL. Then IP is the side of required square equal to the given rectangle ABCD. If we denote $|AB| = a$ and $|BC| = b$, the following identity is used $\left(\frac{b+a}{2}\right)^2 - \left(\frac{b-a}{2}\right)^2 = ab$.

All Sulbasutras contain a method explaining how to square the circle. It is an approximate method based on constructing a square of side $\frac{13}{15}$ times the diameter of the given circle. The result corresponds to $\pi = 4(\frac{13}{15})^2 = \frac{676}{225} \doteq 3.00444$. It is not a very good approximation and certainly not as good as that one known earlier to the Babylonians.

Many different values of π appear in the Sulbasutras, even several different ones used in one text. It is not surprising because the authors thought in terms of approximate constructions, not in terms of exact constructions with π. A given approximate construction implied some value of π.

The Sulbasutras also examine the opposite problem of finding a circle equal in area to the given square. The following construction appears.

Given a square ABCD find the centre S. Rotate SA to the position SP such that SP is perpendicular to the side AB. The point O is the midpoint of the side AB. Let Q be the point on PO such that OQ is one third of OP. The required circle has centre S and radius SQ.

If we denote a the side of the square ABCD, then the diameter of required circle is $d = \left(1 + \frac{\sqrt{2}-1}{3}\right) a$ and the corresponding value of π is $\pi \doteq 3.088$.

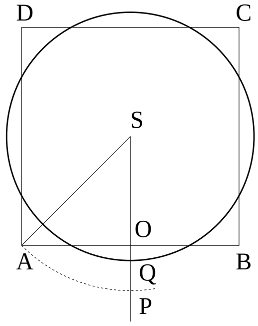

Figure 3. The circle equal in area to the given square.

The most remarkable result of the mathematics of the Sulbasutras is a close approximation to $\sqrt{2}$. Both the Apastamba Sulbasutra and the Katyayana Sulbasutra give the following [*O'Connor, Robertson*]:

> *Increase a unit length by its third and this third by its own fourth less the thirty-fourth part of that fourth.*

Now this gives $\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34} = \frac{577}{408} \doteq 1.414215686$.

Compare the correct value $\sqrt{2} \doteq 1.414213562$ to see that the Apastamba Sulbasutra has the answer correct to five decimal places. (The Babylonians used the value $\sqrt{2} \doteq 1.414212963$ in the 2^{nd} millennium BC [*Bečvář et al.*]).

An early method for calculating square roots can be found in some Sulbasutras [*O'Connor, Robertson*. The method involves repeated application of the formula $\sqrt{Q} = \sqrt{(A^2 + b)} \doteq A + \frac{b}{2A}$, where $A, Q \in N$ and $A^2 < Q < (A+1)^2$.

Advanced numerical calculations required the correct number expression. The definite appearance of decimal symbols for numerals and a place value system is contained in the Vedic mathematics.

Jainism

The Vedic religion with its sacrificial rites began to be replaced by other religions. One of these was Jainism, a religion and philosophy which was founded in India around the 6^{th} century BC. The Jaina religion became the prominent religion in the Indian subcontinent and gave rise to Jaina mathematics. The main Jaina works on mathematics date from around 300 BC to 400 AD.

There are several significant Jaina works including the *Surya Prajinapti* and several Sutras There is also evidence of individual mathematicians including *Bhadrabahu* (possibly lived around 300 BC) and *Umaswati* (possibly lived around 150 BC). Umaswati is known as a great writer on Jaina metaphysics but he also wrote a work *Tattwarthadhigama-Sutra Bhashya* which contains mathematics. Amongs the mathematical results are mensuration formulas which include a circumference of a circle, an area of a circle, a diameter, a chord, a height of the segment. Knowledge of solution to quadratic equations is shown in these formulas.

The Jaina's cosmological ideas influenced mathematics in many ways. The Jainas were fascinated with large numbers, their cosmology contained a time period of 2^{588}years . Calculations with great numbers led to the decimal place value system of numeration and arithmetic developed according to it. The Jaina works refer to a very large number of names giving the positions in the numeral system. The Jainas required very large numbers for their measurements of space and time. The introduction of such large numbers was the impulse to the conception of infinity. Numbers were classified as enumerable, unenumerable and infinite. Infinity itself was of five kinds: infinite in one direction, infinite in two directions, infinite in area, infinite everywhere, infinite perpetually.

Jaina works contain simple laws of indices.

> *The first square root multiplied by the second square root, or the cube of the second square root.*

Expressed in symbols it means $(\sqrt{a}) \cdot (\sqrt[4]{a}) = (\sqrt[4]{a})^3$.

The notation of permutations and combinations has appeared in the Jaina works. The Jaina name for the subject of permutations and combinations is *vikalpa*. Simple problems are solved, such as the number of selections that can be made out of a given number of men and women. Correct formulas for both permutations and combinations are found in Jaina works (for $n \in N$)

$$C_1(n) = n, \quad C_2(n) = \frac{n(n-1)}{1 \cdot 2}, \quad C_3(n) = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3},$$

$$P_1(n) = n, \quad P_2(n) = n(n-1), \quad P_3(n) = n(n-1)(n-2).$$

The method of finding the number of combinations is called *Meru Prastara* and it is a formation of an early Pascal triangle. The Meru Prastara rule is based on the following formula $C_r(n+1) = C_r(n) + C_{r-1}(n)$. A commentator of the 10$^{\text{th}}$ century AD explained it as follows.

> *First draw a square. Below it, and starting from the middle of the low side, draw two squares. Similarly, draw three squares below these, and so on. Write the number 1 in the middle of the top square and inside the first and last squares of each row. Inside every other square, the number to be written is the sum of the numbers in the two squares above it overlapping it.*

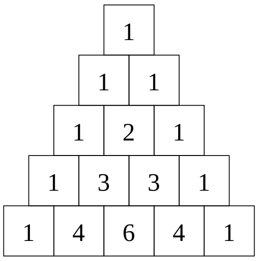

Figure 4. The diagram Meru Prastara.

Bakhshali manuscript

The Bakhshali manuscript is an early mathematical work written on birch bark which was discovered in the summer of 1881 near the village Bakhshali. The most probable date of its origin ranges from 200 to 400 AD. The notation used have features not found in any other document. Fractions are not dissimilar in notation to that used today, written with one number below the other. No line appears between the numbers as we would write today, however. Another unusual feature is the sign + placed after a number to indicate a negative. It is very strange for us today to see our addition symbol being used for subtraction.

For instance $\frac{3}{4} - \frac{1}{2}$ was written [*Datta, Singh*]:

3	1+
4	2

Equations are given with a large dot representing the unknown. A confusing aspect of Indian mathematics is that this notation was also often used to denote zero, and sometimes this same notation for both zero and the unknown are used in the same document. Here is an example of an equation as it appears in the Bakhshali manuscript [*Kaye*].

•	5	*yu*	*mū*	•	*sa*	•	7+	*mū*	•
1	1			1		1	1		1

Yu (yuta) means add, *sa* means and, *mū* (*mūla*) means root (square root). This is in our notation

$$\begin{aligned} x + 5 &= y^2, \\ x - 7 &= z^2. \end{aligned}$$

The method of equalisation is used in many problem solving tasks which occur in the manuscript. The problems concerning equalising wealth, position of two travellers, wages, and purcheses by a number of merchants are included. These problems can all be reduced to solving a linear equation with one unknown or to a system of n linear equations with n unknowns. Some of these problems lead to indeterminate equations.

Another interesting piece of mathematics in the manuscript concerns calculating square roots [*O'Connor, Robertson*].

> *In the case of a non-square number, subtract the nearest square number, divide the remainder by twice this nearest square; half the square of this is divided by the sum of the approximate root and the fraction. This is subtracted and will give the corrected root.*

This means that the following formula is used for $A, Q \in N$ and $A^2 < Q < (A+1)^2$

$$\sqrt{Q} = \sqrt{(A^2 + b)} \doteq A + \frac{b}{2A} - \frac{(\frac{b}{2A})^2}{2(A + \frac{b}{2A})}.$$

The Bakhshali manuscript is a unique piece of work. The method of the commentary follows a highly systematic order. It starts with the statement of rule followed by examples and demonstration of the operation of the rule. Other Indian works were written in a poetic form comprising of short statements of rules, and rarely included examples. This poetic form was favoured because of the limited supplies of writing equipment available.

Conclusion

By about 500 AD the classical era of Indian mathematics began with the work of *Aryabhata*. His work was both a summary of Jaina mathematics and the beginning of new era for astronomy and mathematics. Aryabhata headed a research centre for mathematics and astronomy at Kusumapura. Another mathematical and astronomical centre was at Ujjain which grew up around the same time as Kusumapura. The most important of mathematicians at this second centre was *Varahamihira* who also made important contributions to astronomy and trigonometry. The next scientist of major importance at the Ujjain school was *Brahmagupta* who lived round the beginning of the 7^{th} century AD. He made one of the most essential contributions to the development of number systems with his remarkable contributions to negative numbers and zero. He made other contributions to the understanding of integer solutions to indeterminate equations and to interpolation formulas invented to aid the computation of sine tables. A contemporary of Brahmagupta who headed the research centre at Ujjain was *Bhaskara I*. Bhaskara I was also a commentator on the mathematics of Aryabhata.

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