WDS'05 Proceedings of Contributed Papers, Part I, 198–202, 2005. *ISBN 80-86732-59-2* *© MATFYZPRESS*

Curves in the History of Mathematics: The Late Renaissance

L. Koudela

Charles University, Faculty of Mathematics and Physics, Prague, Czech Republic.

Abstract. Interest in ancient Greek knowledge increased gradually in the fifteenth and sixteenth centuries, especially in Italy, the leading country of Europe's culture and science of the time. Latin translations of Greek works on conic sections and other curves – Apollonius and Pappus in particular – appeared in several editions. Although some original works were also published in the sixteenth century, no significant progress in the study of conic sections had been made until the work of Kepler. His contribution influenced the further development of projective geometry and can be regarded as the transition from ancient to modern geometry.

The spread of Greek knowledge in the Renaissance

The invention of conic sections is attributed to Menaechmus (4th cent. BC), a member of Plato's Academy at Athens. Various species of conic sections were obtained by truncating an acute-angled, right-angled and obtuse-angled cone by a plane perpendicular to the generator of the cone. Conic sections were also investigated by Aristaeus the Elder (4th cent. BC) and by Euclid (c. 325 – c. 265 BC). Their works on this subject are now lost. The works of Archimedes (287 – 212 BC) contain some important results concerning the properties of conic sections, especially parabolas. The greatest ancient writer on conic sections was Apollonius of Perga (c. 262 – c. 190 BC). His famous work Conics consisted of eight books and contained 487 propositions. Apollonius introduced the terms ellipse, parabola and hyperbola and showed that various sections of the cone can be obtained by varying the inclination of the intersecting plane. Among other ancient authors dealing with conic sections, we should mention Pappus of Alexandria (c. 290 – c. 350 AD), the last of great Greek geometers. His main work known as Collection is valuable – among other things – because it provides an account and comments on the results obtained by his predecessors. Pappus introduced the notion of the focus and the directrix of a hyperbola [*Kline, 1990, p. 128*].

The history of conic sections is almost blank since the days of Pappus until the fifteenth century [*Coolidge, 1945, p. 26*]. Renaissance brought a revival of interest in Greek knowledge and consequently in conic sections and other curves. The first four books of Apollonius' Conics survived in Greek, three more in Arabic translation and the eighth book was lost. Pappus' Collection written originally in eight books had been preserved only in the incomplete form.

The Latin translation of the first four books of Apollonius by Gianbattista Memo appeared in Venice in 1537 [*Apollonius, 1537*]. It became the source for Francisco Maurolico of Messina (1494 – 1575) who was preparing the edition of Apollonius' Conics and attempted also to reconstruct the fifth and the sixth books unavailable in Greek. The Arabic translation of books V-VII was discovered in the 17th century. It was translated from Arabic by Abraham Echellensis (d. 1664) and published by Giacomo Alfonso Borelli (1608 – 1679) [*Apollonius, 1661*]. The most notable printed edition of Apollonius' Conics in the sixteenth century is based on the translation by Federico Commandino (1506 – 1575) [*Apollonius, 1566*], who played the key role in the project of editing, translating and publishing classical Greek mathematical texts under the auspices of the Duke of Urbino. The first Commandino's published work was his translation of Archimedes' works including Quadrature of the Parabola and On Spirals [*Archimedes, 1558*]. Pappus' Collection translated also by Federico Commandino was published by Commandino's pupil Guidobaldo del Monte (1545 – 1607) in 1588 [*Pappus, 1588*].

The first original work on conic sections in Christian Europe entitled Libellus super viginti duobus elementis conicis (Book on Twenty-two Elements of Conics) was written by Johannes Werner (1468 – 1528) as early as 1522. He deals with problems derived from those already treated by various Greek authors. It is noteworthy that he writes about the parabola and the hyperbola only. The reason why he makes no mention of the ellipse seems to be that he was primarily interested in the duplication of the cube and the ellipse has no importance in that matter [*Cantor, 1892, p. 419*].

There were also some unusual curves known already in ancient Greece: quadratrix of Hippias, spiral of Archimedes, conchoid of Nicomedes, cissoid of Diocles. They were obtained as means to solve the classical problems of Greek mathematics (squaring the circle, duplicating the cube and trisecting an angle). No other curve was introduced in the Renaissance period, with the eventual exception of the cycloid. According to John Wallis, the cycloid was invented by Nicholas of Cusa in the middle of the fifteenth century [*Cantor, 1892, p.185*]. However, the prominent role, which the cycloid played in mathematics, started as late as in the seventeenth century.

New motivation for the study of curves

Apart from the search of ancient wisdom, new interest in geometry was raised by the application of geometrical principles in art. The mathematical system of perspective of the time was based on the work of Leone Battista Alberti (1404 – 1472) and developed by a number of artists [*Kline, 1990, p. 232*]. The properties of conic sections were also studied in this context. Conic sections received further attention with the study of various optical problems including the function of lenses and mirrors.

The greatest scientific achievement of the Renaissance was undoubtedly the revolution in astronomy. The classical work of Nicholas Copernicus (1473 – 1543) De Revolutionibus Orbium Coelestium (On the Revolution of the Heavenly Spheres) was published in 1543 and introduced the conception of the earth moving around the sun [*Copernicus, 1543*]. The theory of Copernicus encountered heavy objections in following decades and was widely accepted as late as in the seventeenth century.

In the geocentric system of Ptolemy, the difference between observed motion of planets and ideal motion in the circle was interpreted by composition of more circular movements of constant speed. The planets were supposed to move in small circular orbits called epicycles, moving along a larger circle called a deferent. In the Copernican system, the circle preserved its privileged position as the only possible path of heavenly bodies. It was Johannes Kepler (1571 – 1630) in his Astronomia nova (New Astronomy), who first recognized the elliptic path of Mars revolving around the sun. Kepler’s discovery raised new motivation in study of conic sections and their properties related to astronomy and mechanics in particular.

The search for new trade routes to Asia and subsequent geographical explorations stimulated profound changes in cartography. The development of cartography was affected by the works of Johannes Werner mentioned above. The most significant new method of mak-making in the Renaissance was invented by Gerhard Kremer, known also as Mercator (1512 – 1594). The Mercator projection is a cylindrical map projection showing parallels and meridians as straight lines. It was particularly useful for navigation because compass courses can be plotted as straight lines.

Kepler’s work on conic sections

In the beginning of his Prague years Kepler wrote Ad Vitellionem Paralipomena, quibus Astronomiae pars Optica Traditur, published in Frankfurt in 1604. Its title is usually shortened to Astronomiae pars Optica (Optical Part of Astronomy). The title refers to the popular manuscript on optics written in the thirteenth century by Vitello (it was published as printed edition in 1544). Kepler’s work is devoted mostly to his own contribution to various optical problems. A part of its fourth chapter is devoted to the conic sections [*Kepler, 1858, pp. 185-188*].

Kepler discusses there five types of conic sections: circle, ellipse, parabola, hyperbola and line and states that one figure can be obtained from the other by continuous change. Line and parabola are two extreme forms of hyperbola; parabola and circle are the extreme forms of ellipse. Parabola stands between infinite sections (hyperbola and line) and finite sections (circle and ellipse). All sections can be regarded as instances of one geometrical figure with different ratio of curvature and straightness.

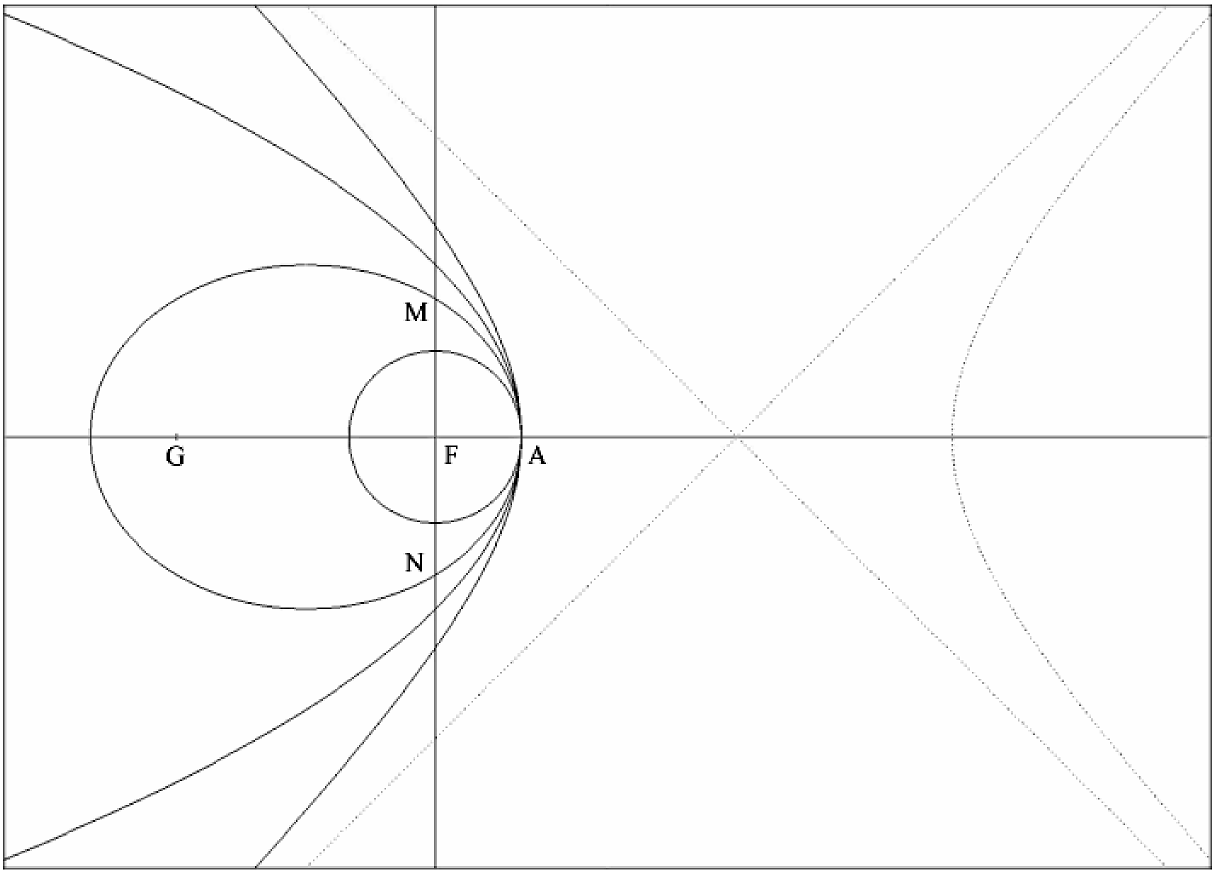

Figure 1. Kepler's idea of continuous transformation of conic sections

Kepler was the first who introduced the name focus for significant points in axis of conic sections. Focus of the circle appears in its centre (F), the ellipse has two foci (F, G) equidistant from its centre (Fig. 1). If we fix the focus F and let the focus G move to infinity, the ellipse becomes a parabola. If the moving focus appears on the other side of the axis (outside of the section), the parabola becomes a hyperbola. If the moving focus approaches again the fixed one, the hyperbola degenerates to the line.

Kepler assumes that all conic sections have two foci. The foci meet each other at a single point in case of the circle and the line. The distance of the foci is greater than zero but finite in ellipse and hyperbola and infinite in parabola. The second focus of the parabola at infinity is called by Kepler the blind (caecus) focus. The line through the blind focus cutting the parabola at any point is parallel to the axis. Kepler uses the term analogy for his principle of continuous transformation of conic sections.

Kepler then examined the focal chord MN perpendicular to the axis and the segment FA of the axis. In our terminology, the length of the semichord FM is the focal parameter p. Let d be the length of FA, i. e. the distance from the focus to the vertex. Kepler states that $p = d$ for the circle, $p > d > p/2$ for the ellipse, $p = 2d$ for the parabola and $p > 2d$ for the hyperbola. The results obtained by Kepler can be expressed together using the eccentricity of a conic section. If ε is the eccentricity, then $p/d = 1 + \varepsilon$.

Table 1. The classification of conic sections.

conic section		eccentricity
circle	$d = p$	$\varepsilon = 0$
ellipse	$p/2 < d < p$	$0 < \varepsilon < 1$
parabola	$d = p/2$	$\varepsilon = 1$
hyperbola	$d < p/2$	$\varepsilon > 1$
line	$d = 0$	$\varepsilon = \infty$

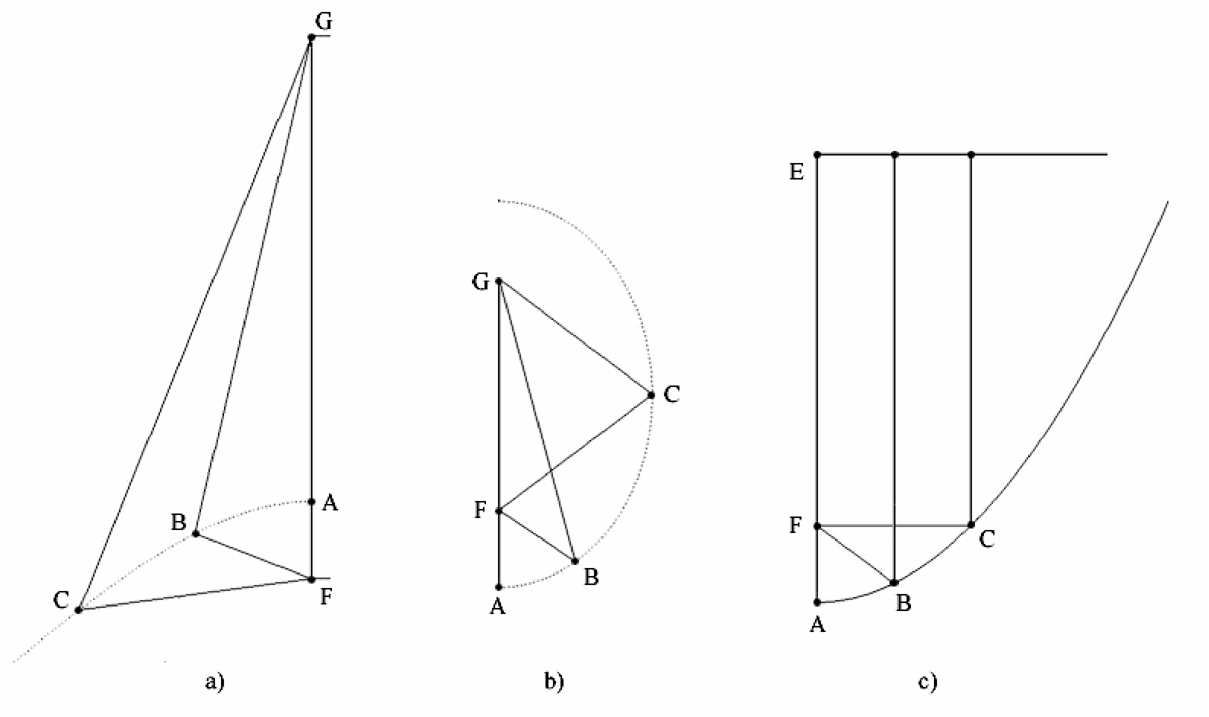

Figure 2. Construction of hyperbola (a), ellipse (b) and parabola (c).

Kepler then describes construction methods for hyperbola, ellipse and parabola using string, pins and pen. In case of hyperbola and ellipse, Kepler refers to Apollonius, book III, 51-52. Let F, G be the foci of hyperbola and A its vertex (Fig. 2a). There are two strings stretched through F resp. G to A. A pen is placed in the common end of both strings in A. If we equally elongate both strings and keep them stretched, then their common end reaches the point B and then C, whereas GA – FA = GB – FB = GC – FC. The path of the pen is then a hyperbola.

The construction of ellipse is even easier. The ends of a string are fixed in two foci F, G, its length is AF + AG, where A is the vertex (Fig. 2b). The ellipse can be drawn, if the pen keeps the string stretched and move around the foci.

For the construction of parabola Kepler employs his own principle of analogy. Let F be the focus, A the vertex and AF the axis. There is a point E on the axis in an arbitrary distance (Fig. 2c). Put the pin in the focus F and stretch the string from F to A and then to E. Place a pen in A. Move the end of the string in E along the line perpendicular to the axis and simultaneously move the pen so that the floating end of the string remains parallel with the axis. Then the pen makes a parabola.

Conclusion

Kepler in his view of the universe stands fully in the Renaissance world. However, his scientific ideas and discoveries belong already to the modern era. The principle of analogy stands in the beginning of search for more general methods in geometry in the seventeenth century. Kepler introduced the idea of transformation of geometrical figure and extended the field of geometry to the infinity. The notion of transformation is connected with the search for properties that remain invariant. It was developed later in more general methods of proof by Blaise Pascal (1623 – 1662), Girard Desargues (1591 – 1661) and others.

The works of Galileo Galilei (1564 – 1642) and Isaac Newton (1643 – 1727) on the laws of motion followed Kepler’s discoveries in astronomy. Introducing conic sections to astronomy also enabled mathematicians to understand curves as continuous paths of moving point or body. It led to the invention of new curves and took part in the formulation of the function concept.

References

Apollonius: *Apollonii Pergei... opera per doctissimum philosophum Joannem Baptistam Memum... de graecon latinum traducta, et noviter impressa*, Venetiis, per Bernardinum Bindonum, 1537.

Apollonius: *Apollonii Pergaei Conicorum libri quattuor, vna cum Pappi Alexandrini lemmatibus, et commentariis Eutocii Ascolonitae. Sereni antinsensis philosophi libri duo nunc primum in lucem editi. Quae omnia nuper Federicus Commandinus*, Bononiae, ex officina alexandri Benatii, 1566.

Apollonius: *Apollonii Pergaei conicorum libri V, VI, VII, paraphraste Abalphato Asphahanensi, nunc primum editi; additus in calce Archimedis assumptorum liber, ex codicibus arabicis manusc. Abrahamus Ecchellensis latinos reddidit; Jo. Alfonsus Borellus curam in geometricis versioni contulit, et notas uberiores in universum opus adjecit*, Florentiae, ex typographia Iosephi Cocchini, 1661.

Archimedes: *Archimedis Opera non nulla a Federico Commandino urbinate nuper in latinum conversa et commentariis illustrata*, Venetiis, apud Paulum Manutium, 1558.

Cantor, Moritz: *Vorlesungen über Geschichte der Mathematik, Bd. 2*, B. G. Teubner, Leipzig, 1892.

Coolidge, Julian Lowell: *A History of the Conic Sections and Quadric Surfaces*, Clarendon Press, Oxford, 1945.

Copernicus, Nicholas: *Nicolai Copernici torinensis de revolutionibus orbium colelestium, Libri VI*, Norimbergae, apud Joh. Petreium, 1543.

Kepler, Johannes: *Joannis Kepleri astronomi Opera omnia, Volumen secundum / ed. Ch. Frisch*, Heyder & Zimmer, Frankofurti a. M. et Erlangae, 1858.

Kline, Morris: *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York, 1990.

Pappus: *Pappi Alexandrini mathematicae collectiones a Federico Commandino Urbinate in latinum conversae at commentariis illustratae*, Pisauri, apud Hieronymum Concordiam, 1588.