State Final Examination (Mathematics Sample Questions)

Summer 2021

1 Derivative (3 points)

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and r a real number. Write the definition of the *derivative* of f at r.

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function that for every $x \in \mathbb{R}$ satisfies $0 \le f(x) \le x^2$. Does this imply that f has a derivative at $x = 0$? Does it imply that $f'(0) = 0$?

3. Let us define a function $q\colon \mathbb{R} \to \mathbb{R}$ by

$$q(x) = \begin{cases} (\sin x)^2 \cdot \left|\sin\left(\frac{1}{x}\right)\right| & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$$

Does q have a derivative at $x = 0$, and if it does, what is its value?

2 Primitive function (3 points)

1. Define the term *primitive function* of a function f on an open interval I.

2. Find a primitive function of the function $f(x) = x^2 \ln x$ on the interval $(0, +\infty)$, where $\ln x$ denotes the natural logarithm.

3. Let k and ℓ be nonnegative integers. Consider the function $f_{k,\ell}(x) = x^k (\ln x)^\ell$ defined on the interval $I = (0, +\infty)$. Show that $f_{k,\ell}$ has a primitive function $F_{k,\ell}$ on I. Moreover, show that this primitive function $F_{k,\ell}$ can be expressed by a formula using constants, the function x, the function $\ln x$, and the operations of addition, subtraction, multiplication, division and exponentiation.

3 Systems of linear equations (3 points)

Let

$$A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 1 & 1 \\ 0 & 2 & 3 \end{pmatrix}.$$

1. Decide whether there exists $b \in \mathbb{R}^3$ such that the system $Ax = b$ has a unique solution.

2. Decide whether there exists $k \ge 1$ such that the system $(A^k)x = b$ has at least one solution for each $b \in \mathbb{R}^3$.

4 Orthogonal projection (3 points)

Let $P \in \mathbb{R}^{n \times n}$ be a matrix of the orthogonal projection onto a real subspace.

1. Simplify the expression $P - P^2 + P^3 - P^4 + \ldots + (-1)^{n+1} P^n$.

2. Prove the following equivalence for each $x \in \mathbb{R}^n$:

$$\|Px\| = \|x\| \quad \Leftrightarrow \quad Px = x.$$

5   Determinants (3 points)

1. Express the Laplace expansion formula for the determinant of a matrix $A \in \mathbb{R}^{9 \times 9}$ along the second column.

2. Decide whether the following equality holds for square matrices of the same order:

$$\det(A + B) = \det(A) + \det(B).$$

Justify your answer (i.e., prove or disprove by counterexample).

6   Trees (3 points)

Let $G = (V, E)$ be a graph with a nonempty set of edges. Decide which of the following conditions is equivalent to G being a tree.

a) G is connected and any two spanning trees of G are isomorphic,

b) G has no cycle, the minimum vertex degree is 1 and $|E| < |V| - 2$,

c) G is connected and has at least two vertices of degree 1 (so-called leaves),

d) G contains a disconnected subgraph, indeed each proper subset of edges $F \subset E$ forms a disconnected graph (V, F), and there is a trail in G between any pair of vertices,

e) in G there is no subset $W \subseteq V$ of vertices that would induce a subgraph with $|W|$ edges, while for every nonempty $W \subsetneq V$ there exists an edge $\{u, v\}$ between W and the complement of W in V.

Justify your conclusions.

7   Binomial coefficients (3 points)

1. Define the binomial coefficient and briefly describe Pascal's triangle.

2. Decide whether for each positive integer n the next two expressions can be compared by the same inequality and if so, determine in which direction the inequality holds.

$$\binom{n}{n} + \binom{n+1}{n} + \binom{n+2}{n} + \cdots + \binom{2n-1}{n} \qquad \text{and} \qquad \binom{2n}{n}.$$

Justify your conclusions. (Hint: use fundamental facts about neighbors in Pascal's triangle.)

8   Expected value (3 points)

1. Define the expected value of a real-valued random variable in a finite probability space Ω.

2. Consider a traditional dice with six sides marked with numbers $1, 2 \ldots, 6$, with each value being equally likely to appear when the dice is rolled. Compute the expected value of the random variable that corresponds to the square of the number observed when the dice is rolled.

3. For $n \in \mathbb{N}$, let S_n be the set of all permutations of the set $\{1, 2, \ldots, n\}$. A fixed point of a permutation $q \in S_n$ is an integer $k \in \{1, 2, \ldots, n\}$ such that $q(k) = k$. Compute the expected value of the number of fixed points of a random permutation (each permutation is selected with the same probability).

9   Logic (3 points)

1. Give a definition for a theory T in a language L_T being a *simple complete extension* of a theory S in a language L_S (for predicate logic).

2. Let $S = \{(\forall x)(\forall y)(\forall z)((U(x) \wedge U(y) \wedge U(z)) \rightarrow (x = y \vee y = z \vee x = z))\}$ be a theory in the language $L = \langle U \rangle$ with equality where U is a unary relation symbol. Write two (nonequivalent) simple complete extensions T_1, T_2 of S.

3. Justify why T_1, T_2 are complete.