State Final Examination (Sample Questions)

Summer 2019

1 Automata and Grammars (3 points)

Let $G = (\{S, A, B\}, \{a, b\}, S, P)$ be a grammar with the set of rules P:

$$\begin{array}{lcl} S & \to & AB \mid \lambda, \\ A & \to & a, \\ AB & \to & BA, \\ B & \to & ABB \mid b, \end{array}$$

where λ is the empty word.

1. Give a clear English description of the language $L(G)$.
2. Design a pushdown automaton accepting the language $L(G)$.
3. Prove that the language $L(G)$ is not regular by applying Nerode theorem or a pumping lemma.

2 Algorithms and data structures (3 points)

1. Define decision problems.
2. Define the classes of problems **P** and **NP**.
3. Define **NP**-completeness of problems.
4. Give an example of an **NP**-complete problem. A proof is not necessary, but please formulate the problem precisely.

3 Databases (3 points)

1. Describe semantics (behavior) of the natural join operation in the relational algebra. What will be the result of a natural join of tables R(A,B)=((0,1),(1,2)) and S(B,C)=((2,3),(3,4))? Is natural join a necessary operation in the relational algebra?
2. Is it useful to support both B-tree based and bitmap-based indexes, or it is enought to use only one of them? Explain your answer.
3. Add a functional dependency to the R(A,B,C) table so that it violates the second normal form.

4 Programming (3 points)

1. Propose an object model that represents a school that has students, teachers, and courses. A course has a name, a unique ID, a teacher, and it is possible to obtain the list of students subscribed for the course. Students and teachers have their names and unique IDs. For each teacher, it is possible to obtain the list of the courses taught. For each student, it is possible to obtain the list of subscribed courses. Warning – a single person can be both a teacher and a student at the same time (for different courses, but it is not necessary to check this).

Follow rules of the object-oriented design. Your model should contain method signatures and fields (attributes), but it is not necessary to include the code for method bodies at this point. Use Java, C# or C++ for the model. Minor syntactic errors will be ignored for grading purposes.

2. Extend your model with an infrastructure (i.e., extend classes as required) for saving the model to a textual file with (roughly) the following structure:

```
people:
  - id: 12345
    name: Alan Doe
  - id: 98765
    name: Jane Smith
  - id: 34567
    name: John Poe
courses:
  - id: NPRG000
    name: Programming
teachers:
  - id: 12345
    teaches: NPRG000, NSWI000
  - id: 34567
    teaches: NTIN000
students:
  - id: 98765
    subscribed: NPRG000, NTIN000
  - id: 34567
    subscribed: NPRG000, NSWI000
```

Focus primarily on the signatures of methods/functions, the method/function bodies can be just sketched. If the programming language of your choice offers constructs for object saving, you can use them but it is not mandatory.

3. If you have used multiple inheritance in your model, very briefly sketch a model created without it. And vice versa, if you have not used multiple inheritance, sketch the model with it. (Note – depending on the chosen programming language, multiple inheritance can exist not only on classes but also e.g. on interfaces or other constructs.) Briefly summarize the advantages and disadvantages of this alternative model.

5 Architectures (3 points)

You have looked up the information on the `.BMP` image file format on Wikipedia. The file starts with this `FILEHEADER`:

Offset hex	Size	Purpose
`00`	2 bytes	Identifies the BMP file, and is `0x42 0x4D` in hexadecimal, same as `BM` in ASCII.
`02`	4 bytes	The size of the BMP file in bytes.
`06`	2 bytes	Reserved; actual value depends on the application that creates the image.
`08`	2 bytes	Reserved; actual value depends on the application that creates the image.
`0A`	4 bytes	The offset of the byte where the bitmap image data (pixel array) can be found.

The `FILEHEADER` is immediately followed by `BITMAPHEADER`:

Offset hex	Size	Purpose
`0E`	4 bytes	The size of this header.
`12`	4 bytes	The bitmap width in pixels (signed integer in two's complement).
`16`	4 bytes	The bitmap height in pixels (signed integer in two's complement).
`1A`	2 bytes	The number of color planes (must be 1).
`1C`	2 bytes	The number of bits per pixel, which is the color depth of the image.

You want to load images in the `.BMP` format into your application. To do that, solve the following:

1. Explain what is usually meant by *data alignment*. Explain whether the presented fragments of the `.BMP` file format are aligned or not (assume a common computer architecture).

2. The `.BMP` file format uses *little endian* encoding for all data. Explain what that means.

3. Using Java or C# or C++, sketch a simple program that will read a `CoolPicture.bmp` file and print its width in pixels on standard output as a base-10 integer. Your program will need to access the file using a file API – it is not required that you use an existing API, just use a reasonale (even if made up) API.

When writing code in your answer, approximate syntax (pseudocode) is enough, provided it is clear what the eventual implementation in the target programming language should look like (e.g. lines do not have to end with statement separators etc.).

6 Networks (3 points)

Write the sequence of sent and received FTP messages, which connect to the `pirate.server.eu`, download the binary file `/hide/movie.mkv`, and disconnect. The FTP server is in passive mode, the port range for data transfers is 10000–10999, the server IP address is 195.113.19.222. We will use `anonymous` user for login and we send email `my@secretmail.edu` instead of password.

In addition to the basic sequence of FTP messages with the necessary parameters, a general description of the session is expected, in particular, an explanation of which transport protocol and how to send individual messages. Minor syntactic errors do not matter in the answer.

7 Optimization (3 points)

1. Consider the linear program (P) given by $\max c^T x$, $Ax \leq b$, $x \geq 0$. Formulate the dual program (D). State and prove the weak duality theorem for the pair (P) and (D) of linear programs.
2. We are given an oriented graph $G = (V, E)$, with $V = \{s, a, b, t\}$ and $E = \{(s, a), (s, b), (a, b), (b, a), (a, t), (b, t)\}$ together with edge capacities $c(s, a) = 1, c(s, b) = 4, c(a, b) = 2, c(b, a) = 2, c(a, t) = 3, c(b, t) = 2$.

   Formulate the maximum $s - t$ flow in G with capacities c as a linear program. By applying the weak duality theorem to your linear program, prove that the size of the maximum flow is not larger than 6.

8 ... (3 points)

9 Codes (specialization question – 3 points)

For a positive integer n, let C be a binary code containing all words $(x_1, \ldots, x_{3n})$ of length $3n$ such that $x_{3i-2} = x_{3i-1} = x_{3i}$ for every $i = 1, 2, \ldots, n$. That is,

$$C = \{(x_1, x_1, x_1, x_2, x_2, x_2, \ldots, x_n, x_n, x_n) \mid (x_1, x_2, \ldots, x_n) \in \{0, 1\}^n\} .$$

Determine the parameters of the code C and, for each n, decide whether C is linear and perfect. Justify your answers.

10 Well ordering (specialization question – 3 points)

1. Write a definition of well ordering. It is not necessary to define the notion of ordering itself.
2. Is every well ordering also linear? Is every linear ordering also well ordering? (Write a short explanation.)
3. Write a statement about well orderings that is equivalent to the axiom of choice.

11 Graph colourings (specialization question – 3 points)

1. Write the definition of the vertex chromatic number and the edge chromatic number (also known as the chromatic index) of a graph.

2. Find all positive integers k for which there is a graph whose edge chromatic number equals 5 and vertex chromatic number equals k.

12 Vectors (3 points)

1. Let $x = (4, 1, 2)^T$ and $y = (1, 0, 3)^T$. Find a nonzero vector v such that it belongs to the linear span of x, y, and in addition v is perpendicular to y.
2. In the previous task, let $y = (1, 0, 3)^T$ be fixed, but x will be a variable vector. Find a formula for v in terms of the unknown x.

   Since this formula represents a linear mapping, find a matrix of this mapping, corresponding to the canonical basis.

13 Groups (3 points)

Let $n \in \mathbb{N}$ be fixed and consider the set $\mathcal{M}$ of all $n \times n$ matrices with integer entries and having the determinant equal to 1. Show that $\mathcal{M}$ equipped with the matrix product forms a group.

14 Matrices (3 points)

Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

Decide whether the matrices

1. have the same determinant;
2. are similar;
3. represent matrices of the same quadratic form, but with respect to possibly different bases.

15 Sequences and limits (3 points)

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and let $\alpha \in \mathbb{R}$. Define the meaning of $\lim_{n\to\infty} a_n = \alpha$.

Find the limits of sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ defined as $a_n = \frac{n+7}{n^2+9}$ and $b_n = \frac{2^n+5}{2^n}$ for all $n \in \mathbb{N}$.

16 Total differential (3 points)

Define the total differential (also known as total derivative). For the function $f(x, y) = x^2 + y^2$ calculate the total differential $Df(x, y)(h_1, h_2)$ in the usual way and then verify your result using the definition.

17 Area and volume (3 points)

For curves $y = x^2 - 2x$ and $y = x$ determine:

1. The area bounded by the curves.
2. The volume of a solid of revolution obtained by rotating the region bounded by the curves around the axis $y = 4$.

18 Graphs (3 points)

State the definition of the edge connectivity and the vertex connectivity of a graph.

For $k \geq 2$, is every k-regular connected graph also vertex 2-connected?

19 Finite projective planes (3 points)

Consider a finite projective plane (FPP) on the set $\{1, 2, \ldots, k\}$. It is known that this FPP contains the lines $\{1, 2, \ldots, 10\}$, $\{10, 11, \ldots, 19\}$ and $\{19, 20, \ldots, 27, 1\}$, besides other lines.

1. Determine the value of k.
2. How many lines of this FPP intersects the set $M = \{1, 11, 21\}$?

20 Logic (3 points)

1. Define when a formula φ is *independent* in a theory T.
2. Consider a (propositional) theory $T = \{p \leftrightarrow \neg(q \rightarrow r),\ \neg(\neg p \wedge q) \vee \neg r\}$ over the set of propositional letters $\mathbb{P} = \{p, q, r\}$. Determine its models.
3. Find an example of formula φ over $\mathbb{P}$ that is independent in the above theory T, or show that such formula does not exist.