State Final Examination (Mathematics Sample Questions)

Spring 2020

1 Series (3 points)

1. Define the convergence of a series.
2. State the necessary condition for convergence of a series. Give an example of a series which satisfies the necessary condition but is not convergent.
3. Decide whether the following series converges and justify your decision.

$$\sum_{n=1}^{\infty} \frac{n^2}{n!}$$

2 Derivative (3 points)

1. Define the derivative of a function at a point. Using the definition, find the derivative of the function $f(x) = \frac{1}{x}$ at a point $a \neq 0$.
2. Decide whether the following statement is true and justify your decision: If f is continuous at the point 1, f is differentiable at 1.
3. Find the derivative of the function $\sqrt[x]{x}$.

3 Function of several variables (3 points)

1. Define the Hessian matrix and state the theorem about its connection to local extrema of a function of several variables.
2. Find all local extrema of the function $f(x, y) = -y^2 + \sin x$ and decide whether they are local minima or maxima.

4 Positive definite matrices (3 points)

Define positive definite matrices and describe two methods for testing positive definiteness.

5 Matrix subspaces (3 points)

For matrix

$$A = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 1 & 4 & -1 & 1 \\ 2 & -1 & 0 & 1 \\ 1 & 0 & 3 & -1 \end{pmatrix},$$

find a basis of the intersection of subspaces U and V, where

- U is defined as the orthogonal complement of the kernel of matrix A and
- V is defined as the column space of matrix A^T.

6 Matrices of linear mappings (3 points)

Decide and justify whether the following two matrices are identical:

- the matrix of a rotation of 90 degrees counter-clockwise in $\mathbb{R}^2$ with respect to the canonical basis and
- the matrix representing the change of basis from basis B_1 to basis B_2, where $B_1 = \{1, x\}$ and $B_2 = \{-x, 1\}$ are two bases of the space of linear real functions in one variable x.

7 Combinatorial counting (3 points)

Let A and B be two disjoint subsets of the set of natural numbers such that $|A| = 10$, $|B| = 15$. How many 7-tuples $(x_1, x_2, \ldots, x_7)$ of numbers satisfy all of the following conditions:

- $|\{x_1, x_2, \ldots, x_7\} \cap (A \cup B)| = 7$,
- $|\{x_1, x_2, \ldots, x_7\} \cap A| = 3$,
- $x_i \in B \wedge x_j \in B \wedge i < j \Rightarrow x_i < x_j$?

(In other words, the 7-tuple consists of 7 distinct numbers, 3 of them are from the set A, and the numbers from the set B are in increasing order.)

8 Spanning trees (3 points)

1. Define a spanning tree of a graph.
2. How many spanning trees does the complete graph on n vertices, K_n, have?
3. Find two non-isomorphic graphs with the same numbers of vertices and edges, both with exactly 6 spanning trees.

9 Logic (3 points)

1. Define when a theory T_1 in a language L_1 is an *extension* of a theory T_2 in a language L_2, and when T_1, T_2 are *equivalent*.
2. Consider the following three theories in propositional logic. Write all pairs picked from the three theories where the former theory is an extension of the latter theory. Show why it is so.
   - $T_1 = \{\neg p \vee \neg q\}$ in the language $\{p, q\}$,
   - $T_2 = \{p \leftrightarrow \neg q\}$ in the language $\{p, q\}$,
   - $T_3 = \{\neg(\neg p \leftrightarrow q) \rightarrow r,\ r \leftrightarrow p \vee q\}$ in the language $\{p, q, r\}$.
3. Determine the number of mutually nonequivalent extensions in $\{p, q, r\}$ of the theory $T_1 \cup T_3$. Give an explanation.