1 Continuous functions (3 points)

1. Define what it means that a function \( f : \mathbb{R} \to \mathbb{R} \) is continuous at a point \( b \in \mathbb{R} \).

2. For each of the two functions below, determine (and justify briefly) whether it is continuous in the point 0.

\[
\begin{align*}
f_1(x) &= \begin{cases} 
    x & \text{for } x \in \mathbb{Q} \\
    -x & \text{for } x \notin \mathbb{Q}.
\end{cases} \\
f_2(x) &= \begin{cases} 
    0 & \text{for } x = 0 \\
    \exp(-1/x) & \text{for } x \neq 0.
\end{cases}
\end{align*}
\]

3. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function satisfying \( 0 \leq f(x) \leq 1 \) for each \( x \in \mathbb{R} \). Suppose that \( f \) is Riemann-integrable on the interval \([0, 1]\), and define the function \( g : [0, 1] \to \mathbb{R} \) by

\[
g(x) = \int_0^x f(t) \, dt.
\]

Is \( g \) continuous on the interval \((0, 1)\)? Justify.

2 Limit of a sequence (3 points)

1. Define what it means that a real number \( L \in \mathbb{R} \) is the limit of a sequence of real numbers \((a_n)_{n=0}^{\infty}\).

2. Let \((a_n)_{n=0}^{\infty}\) be a sequence of real numbers with a limit \( L \in \mathbb{R} \). Let us define a sequence \((b_n)_{n=0}^{\infty}\) by the formula

\[
b_n = a_n - a_{2n}.
\]

Can we deduce from this information whether \((b_n)_{n=0}^{\infty}\) has a limit, and determine the value of the limit if it exists?

3. Define a sequence \((c_n)_{n=0}^{\infty}\) by these identities:

\[
c_0 = 1 \\
c_n = \sin(c_{n-1}) \quad \text{for } n \geq 1.
\]

Does \((c_n)_{n=0}^{\infty}\) have a limit? If it does, what is the limit’s value?

3 Primitive function (3 points)

1. Define the notion of primitive function (a.k.a. antiderivative) to a function \( f \) on an interval \((a, b)\).

2. For each of the next two statements, determine whether it is true or false. Justify briefly your answers.

(a) If a function \( f \) is nondecreasing on an interval \([a, b]\), then \( f \) has a primitive function on \((a, b)\).

(b) If a function \( f \) has a primitive function \( F \) on an interval \((a, b)\), and if \( F \) has a local minimum in a point \( c \in (a, b) \), then \( f(c) = 0 \).

3. Compute

\[
\int_{\pi/2}^{\pi} \sqrt{\sin x \cdot \cos x} \, dx.
\]
4 Linear mappings (3 points)

1. Define the kernel $\text{Ker}(f)$ of a linear map $f$ between vector spaces $U$ and $V$.
2. Prove that the kernel $\text{Ker}(f)$ forms a subspace of $U$.
3. Find a basis of the kernel of the linear map that represents the second derivative on the space of real polynomials of degree at most 5.

5 Inner product (3 points)

Consider space $\mathbb{R}^3$ and two inner products $\langle x, y \rangle = x^Ty$, $\langle x, y \rangle_A = x^T Ay$, where

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
1 & 2 & 0 \\
1 & 0 & 3
\end{pmatrix}.
\]

1. Prove that $\langle x, y \rangle_A$ indeed forms an inner product (it is sufficient to show the property $\langle x, x \rangle_A > 0$ for each $x \neq 0$).
2. Find a linear map $f: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that for each $x, y \in \mathbb{R}^3$ we have $\langle f(x), f(y) \rangle = \langle x, y \rangle_A$.

6 Eigenvalues (3 points)

Consider matrices

\[
A = \begin{pmatrix}
3 & 3 & 3 \\
0 & 2 & 2 \\
0 & 0 & 1
\end{pmatrix}, \quad B = \begin{pmatrix}
1 & 1 & 0 \\
0 & 2 & 0 \\
0 & 3 & 3
\end{pmatrix}.
\]

1. Define matrix similarity.
2. Decide whether matrices $A, B$ are similar and justify your answer.
3. Prove $\lim_{n \rightarrow \infty} \left( \frac{1}{4} A \right)^n = \lim_{n \rightarrow \infty} \left( \frac{1}{4} B \right)^n$.

7 Combinatorial counting (3 points)

How many different square matrices of order 4 that contain 9 zeros, but no row or column is completely filled with zeros, can be found above the field $\mathbb{F}_3$?

8 Graph connectivity (3 points)

For a simple graph $G$ with at least two vertices define the notion of the vertex connectivity and the edge connectivity.

Decide whether any graphs with the following parameters exist.

1. $k_v(G_1) = 2, k_e(G_1) = 3$,
2. $k_v(G_2) = 3, k_e(G_2) = 2$,
3. $k_v(G_3) = 2, k_e(G_3) = 3$ and $G_3$ is a cubic graph (i.e. all its vertices are of degree 3).

9 Logic (3 points)

1. Give a definition (one of mutually equivalent definitions) when a theory $S$ is a simple extension of a theory $T$.
2. Find an example of propositional theories $S, T_1, T_2$ such that $S$ is a simple consistent extension both of $T_1$ and of $T_2$ and $T_1 \cup T_2$ is inconsistent (i.e. contradictory), or show why such theories do not exist.
3. What is the number of mutually nonequivalent simple extensions $S$ of the propositional theory $T = \{ p \rightarrow q \}$ over the set of letters $\mathbb{F} = \{ p, q, r \}$ in which $q \rightarrow r$ does not hold, i.e., $S \not\models q \rightarrow r$?