State Final Examination (Mathematics Sample Questions)

Fall 2021

1 Continuous functions (3 points)

1. Define what it means that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *continuous* at a point $b \in \mathbb{R}$.
2. For each of the two functions below, determine (and justify briefly) whether it is continuous in the point 0.

$$f_1(x) = \begin{cases} x & \text{for } x \in \mathbb{Q} \\ -x & \text{for } x \notin \mathbb{Q}. \end{cases}$$

$$f_2(x) = \begin{cases} 0 & \text{for } x = 0 \\ \exp(-1/x) & \text{for } x \neq 0. \end{cases}$$

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function satisfying $0 \leq f(x) \leq 1$ for each $x \in \mathbb{R}$. Suppose that f is Riemann-integrable on the interval $[0, 1]$, and define the function $g\colon [0, 1] \to \mathbb{R}$ by

$$g(x) = \int_0^x f(t)\, dt.$$

Is g continuous on the interval $(0, 1)$? Justify.

2 Limit of a sequence (3 points)

1. Define what it means that a real number $L \in \mathbb{R}$ is the *limit* of a sequence of real numbers $(a_n)_{n=0}^{\infty}$.
2. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers with a limit $L \in \mathbb{R}$. Let us define a sequence $(b_n)_{n=0}^{\infty}$ by the formula $b_n = a_n - a_{2n}$. Can we deduce from this information whether $(b_n)_{n=0}^{\infty}$ has a limit, and determine the value of the limit if it exists?
3. Define a sequence $(c_n)_{n=0}^{\infty}$ by these identities:

$$\begin{aligned} c_0 &= 1 \\ c_n &= \sin(c_{n-1}) \qquad \text{for } n \geq 1. \end{aligned}$$

Does $(c_n)_{n=0}^{\infty}$ have a limit? If it does, what is the limit's value?

3 Primitive function (3 points)

1. Define the notion of *primitive function* (a.k.a. *antiderivative*) to a function f on an interval (a, b).
2. For each of the next two statements, determine whether it is true or false. Justify briefly your answers.
   (a) If a function f is nondecreasing on an interval $[a, b]$, then f has a primitive function on (a, b).
   (b) If a function f has a primitive function F on an interval (a, b), and if F has a local minimum in a point $c \in (a, b)$, then $f(c) = 0$.
3. Compute

$$\int_{\pi/2}^{\pi} \sqrt{\sin x} \cdot \cos x\, dx.$$

4 Linear mappings (3 points)

1. Define *the kernel* $Ker(f)$ of a linear map f between vector spaces U and V.
2. Prove that the kernel $Ker(f)$ forms a subspace of U.
3. Find a basis of the kernel of the linear map that represents the second derivative on the space of real polynomials of degree at most 5.

5 Inner product (3 points)

Consider space $\mathbb{R}^3$ and two inner products

$$\langle x, y\rangle = x^T y,$$
$$\langle x, y\rangle_A = x^T Ay,$$

where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 3 \end{pmatrix}.$$

1. Prove that $\langle x, y\rangle_A$ indeed forms an inner product (it is sufficient to show the property $\langle x, x\rangle_A > 0$ for each $x \neq 0$).
2. Find a linear map $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that for each $x, y \in \mathbb{R}^3$ we have

$$\langle f(x), f(y)\rangle = \langle x, y\rangle_A\,.$$

6 Eigenvalues (3 points)

Consider matrices

$$A = \begin{pmatrix} 3 & 3 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 3 & 3 \end{pmatrix}.$$

1. Define *matrix similarity.*
2. Decide whether matrices A, B are similar and justify your answer.
3. Prove $\lim_{n\to\infty}(\frac{1}{4}A)^n = \lim_{n\to\infty}(\frac{1}{4}B)^n$.

7 Combinatorial counting (3 points)

How many different square matrices of order 4 that contain 9 zeros, but no row or column is completely filled with zeros, can be found above the field $\mathbb{Z}_3$?

8 Graph connectivity (3 points)

For a simple graph G with at least two vertices define the notion of the vertex connectivity and the edge connectivity.

Decide whether any graphs with the following parameters exist.

1. $k_v(G_1) = 2, k_e(G_1) = 3$,
2. $k_v(G_2) = 3, k_e(G_2) = 2$,
3. $k_v(G_3) = 2, k_e(G_3) = 3$ and G_3 is a cubic graph (i.e. all its vertices are of degree 3).

9 Logic (3 points)

1. Give a definition (one of mutually equivalent definitions) when a theory S is a *simple extension* of a theory T.
2. Find an example of propositional theories S, T_1, T_2 such that S is a simple consistent extension both of T_1 and of T_2 and $T_1 \cup T_2$ is inconsistent (i.e. contradictory), or show why such theories do not exist.
3. What is the number of mutually nonequivalent simple extensions S of the propositional theory $T = \{p \to q\}$ over the set of letters $\mathbb{P} = \{p, q, r\}$ in which $q \to r$ does not hold, i.e., $S \not\models q \to r$?