State Final Examination (Mathematics Sample Questions)

Fall 2019

1 Limits of functions (3 points)

1. Define the limit of a function at a point.
2. Let f be a function defined on $\mathbb{R}$. Consider the following two statements P and Q:
   P: $\lim_{x\to\infty} f(x) = 0$
   Q: A sequence $(a_n)_{n=1}^{\infty}$ defined as $a_n = f(n)$ for every $n \in \mathbb{N}$ has limit zero.
   Decide whether P implies Q and whether Q implies P if:
   (a) f is any function,
   (b) f is a monotonic function,
   (c) f is a continuous function.

   You do not need to justify your decision.
3. Find the limit
$$\lim_{x\to\pi/2} \frac{1-\sin x}{1+\cos 2x} .$$

2 Definite integral (3 points)

1. State the substitution theorem for definite integral.
2. Evaluate the following integral:
$$\int_1^2 \frac{3^x+2}{3^{2x}+3^x}\, dx$$
3. Decide which of the following interval contains the numerical value of the integral: $(-\infty, 0]$, $(0, 1/2]$, $(1/2, 1]$,$(1, 3]$, $(3, \infty]$. (This can be determined without evaluating the integral – errors in the previous calculations are not an excuse for an incorrect answer.)

3 Metric spaces (3 points)

Let (X, d) be a metric space, $|X| \geq 2$.

1. Decide whether (X, d') is a metric space for d' defined as
$$d'(x, y) = \min(d(x, y), 1)$$
   for all $x, y \in X$. Justify your answer.
2. Distance between two nonempty subsets A and B of the metric space (X, d) is defined as
$$D(A, B) = \inf_{a\in A, b\in B} d(a, b).$$
   Show that D **is not** a metric on $\mathcal{P}(X) \setminus \{\emptyset\}$ (where $\mathcal{P}(X)$ denotes the power set of X).

4 Linear mappings (3 points)

Let $f, g\colon U \to V$ be linear mappings (homomorphisms) between vector spaces U, V. Decide whether the following statements are true. Explain your answers.

1. $Ker(f) \cap Ker(g) \subseteq Ker(f+g)$,
2. $f(U) + g(U) = (f+g)(U)$.

5 Cauchy–Schwarz inequality (3 points)

1. Define the term "the norm induced by an inner product".
2. Formulate the Cauchy–Schwarz inequality.
3. Prove that both sides of the Cauchy–Schwarz inequality are equal for linearly dependent vectors.

6 Random graphs (3 points)

Assume a random graph on the vertex set $\{1, 2, \dots, 100\}$ with edge probability $1/3$ independetly for each edge.

1. What is the probability that a (randomly choosen) 4-tuple of vertices induces a subgraph isomorphic to C_4?
2. What is the probability that the subgraph induced by vertices $\{10, 20, 30, 40\}$ is isomorphic to C_4 if we know that the pairs of vertices $\{10, 30\}$ and $\{20, 40\}$ do not form edges?
3. What is the expected value of the number of induced copies of C_4?

It is not necessary to evaluate the answers numerically. Factorials, fractions, binomial coeficients etc. are accepted.

7 Binary relations (3 points)

1. For a binary relation $f \subseteq X \times X$ define when f is reflexive, symmetric and transitive.
2. Decide which of these properties are satisfied for $f_1 \subseteq \mathbb{N} \times \mathbb{N}$ defined as

$$(a, b) \in f_1 \Leftrightarrow a, b \text{ are not relative primes.}$$

3. Decide which of these properties are satisfied for $f_2 \subseteq \mathbb{N} \times \mathbb{N}$ defined as

$$(a, b) \in f_2 \Leftrightarrow a \text{ divides } b.$$

8 Spanning trees (3 points)

Let G be a graph with the vertex set $\{u_1, \dots, u_{100}, v_1, \dots, v_5\}$ such that vertices $u_1, \dots, u_{100}$ induce a complete graph, vertices $v_1, \dots, v_5$ form a cycle (in this order), and u_1, v_1 are joined by an edge. There are no other edges in G.

The weight of any edge $u_i u_j$ is equal to 5 for all $1 \leq i < j \leq 100$, the weight of any edge $v_i v_j$ is equal to $\max\{i, j\}$ for $j = (i \bmod 5) + 1$, and the weight of the edge $u_1 v_1$ is equal to 10.

1. Define the spanning tree of a graph and the minimum spanning tree (together with the assumptions for the minimum spanning tree problem).
2. Evaluate the weight of the minimum spanning tree of G.
3. How many minimum spanning trees of G exist?

9 Logic (3 points)

1. Write a definition of a *consistent theory* (in predicate logic).
2. Express the following statements as propositional formulas φ_1, φ_2, φ_3 over $\mathbb{P} = \{r, s, t\}$ where the letters r, s, t represent (resp.) that *"Radka / Sarah / Tom is at school"*.
   (a) *If Tom is not at school, Sarah is not there either.*
   (b) *Radka does not go to school without Sarah.*
   (c) *If Radka is not at school, Tom is there.*
3. Determine whether the theory $T = \{\varphi_1, \varphi_2, \varphi_3\}$ is consistent by using the tableau method, an implication graph, or a resolution closure. That is, it is not enough to only try the individual truth assignments.