# Curves on the torus with prescribed intersections

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- *Approximation algorithms for finding as many disjoint cycles as possible from a certain family of cycles in a given planar or bounded-genus graph.*

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- *Approximation algorithms for finding as many disjoint cycles as possible from a certain family of cycles in a given planar or bounded-genus graph.*
- Question: Given a genus $g$ surface and $k > 0$, what is the maximum number of simple curves that are non-homotopic and they intersect pairwise at most $k$-times?
- Let $\Sigma$ be a compact surface and $k \in \mathbb{N}$, the set of simple curves that are non-homotopic and they intersect pairwise at most $k$-times is called a $k$-system.
- Question: What is the maximum size $N(\Sigma, k)$ of a $k$-system?

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- For $k = 1$, $k$-systems of size $O(g^2)$ have been constructed.
- Is the result tight?

# Can we improve on Greene?

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  - $N(\mathbb{T}^2, k) \leq (1 + o(1))k$
  - Baker, Harman and Pintz via prime number gap: $N(\mathbb{T}^2, k) \leq k + O(k^{21/40})$.

# Let's look at the Torus some more

**Cramér in 1920:**

If the Riemann hypothesis holds, then

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and under further number-theoretic conjecture even*
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Observations:

- combinatorial + geometric arguments + estimates from analytic number theory
- Is there a $k$-system on the torus whose size exceeds $k+6$?

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**Balla, F, Kiełak, Kráľ and Schlomberg**

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**Theorem**

*For every $k \in \mathbb{N} \setminus K_0$, it holds that*

$$N(\mathbb{T}^2, k) = \begin{cases} k+4 & \textit{if } k \bmod 6 = 2, \\ k+3 & \textit{if } k \bmod 6 \in \{1,3,5\}, \textit{and} \\ k+2 & \textit{otherwise.} \end{cases}$$

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*$K_0$ is a special set containing 59 integers.*

# $K_0$

| $k$ | 1 | 2 | 19 | 23 | 24 | 25 | 33 | 34 | 37 | 47 |
|---|---|---|---|---|---|---|---|---|---|---|
| $N(\mathbb{T}^2, k)$ | 3 | 4 | 23 | 27 | 30 | 30 | 37 | 38 | 42 | 51 |
| $N(\mathbb{T}^2, k) - k$ | +2 | +2 | +4 | +4 | +6 | +5 | +4 | +4 | +5 | +4 |
| "pattern" | +3 | +4 | +3 | +3 | +2 | +3 | +3 | +2 | +3 | +3 |

| $k$ | 48 | 49 | 53 | 54 | 55 | 61 | 62 | 63 | 64 | 76 |
|---|---|---|---|---|---|---|---|---|---|---|
| $N(\mathbb{T}^2, k)$ | 54 | 54 | 57 | 59 | 60 | 65 | 67 | 67 | 68 | 80 |
| $N(\mathbb{T}^2, k) - k$ | +6 | +5 | +4 | +5 | +5 | +4 | +5 | +4 | +4 | +4 |
| "pattern" | +2 | +3 | +3 | +2 | +3 | +3 | +4 | +3 | +2 | +2 |

| $k$ | 83 | 84 | 85 | 89 | 90 | 94 | 113 | 114 | 115 | 118 |
|---|---|---|---|---|---|---|---|---|---|---|
| $N(\mathbb{T}^2, k)$ | 87 | 89 | 89 | 93 | 94 | 98 | 117 | 119 | 119 | 122 |
| $N(\mathbb{T}^2, k) - k$ | +4 | +5 | +4 | +4 | +4 | +4 | +4 | +5 | +4 | +4 |
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| $k$ | 119 | 120 | 121 | 124 | 127 | 139 | 141 | 142 | 143 | 144 |
|---|---|---|---|---|---|---|---|---|---|---|
| $N(\mathbb{T}^2, k)$ | 123 | 126 | 126 | 128 | 132 | 143 | 145 | 147 | 147 | 149 |
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| "pattern" | +3 | +2 | +3 | +2 | +3 | +3 | +3 | +2 | +3 | +2 |

| $k$ | 145 | 154 | 167 | 168 | 169 | 174 | 184 | 204 | 208 | 214 |
|---|---|---|---|---|---|---|---|---|---|---|
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| $k$ | 234 | 244 | 264 | 274 | 294 | 304 | 324 | 354 | 384 | |
|---|---|---|---|---|---|---|---|---|---|---|
| $N(\mathbb{T}^2, k)$ | 238 | 247 | 268 | 277 | 297 | 307 | 327 | 357 | 387 | |
| $N(\mathbb{T}^2, k) - k$ | +4 | +3 | +4 | +3 | +3 | +3 | +3 | +3 | +3 | |
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- Every non-trivial closed curve $C$ is freely homotopic to a curve $C_{m,n}$ for some non-zero $(m,n) \in \mathbb{Z}^2$
- if the curve is non-self-intersecting, i.e. simple, then $m$ and $n$ are coprime.
- minimal number of crossings between curves?

# More geometric observations in $\mathbb{T}^2$

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Conclusion:

- We can represent a $k$-system $K$ by a list of pairs of coprime integers $Q(K)$.
- $\Gamma \cdot Q(K)$ represents the same $k$-system for any $\Gamma \in \mathrm{SL}(2, \mathbb{Z})$.

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Conclusion:

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We can use this to put all the points in a *nice position*

# Formally

For an integer $k \in \mathbb{N}$, a set $Q \subseteq \mathbb{Z}^2$ is *k-nice* if the following holds:

- $Q$ contains non-zero coprime pairs only,
- $Q$ does not contain both $(x, y)$ and $(-x, -y)$ for any $(x, y) \in \mathbb{Z}^2$, and
- $|xy' - x'y| \leq k$ for all $(x, y)$ and $(x', y')$ contained in $Q$.

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**Lemma**

*For every $k \in \mathbb{N}$, $N(\mathbb{T}^2, k)$ is equal to the maximum size of a k-nice set.*

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**Lemma**

*For every $k \in \mathbb{N}$, $N(\mathbb{T}^2, k)$ is equal to the maximum size of a k-nice set.*

- $Q \subseteq \mathbb{Z}^2$ is *y-non-negative* if $y \geq 0$ for all $(x, y) \in Q$.
- The *height* of $Q \subseteq \mathbb{Z}^2$ is the maximum $h \in \mathbb{N}$ such that the set $Q$ contains $(x, y)$ with $|y| = h$

# What is the volume of the convex hull of Q?

**Lemma**

*Let Q be k-nice, $y \geq 0$. If $(m, n) \in \mathbb{Z}^2$ is contained in the convex hull of Q and the integers m and n are coprime, then $(m, n)$ is contained in Q.*

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- The density of coprime numbers in $\mathbb{Z}^2$ is roughly $\frac{6}{\pi}$
- What is the volume of conv$(Q)$?

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- The density of coprime numbers in $\mathbb{Z}^2$ is roughly $\frac{6}{\pi}$
- What is the volume of $\mathrm{conv}(Q)$?

**Lemma**

*Let $k \in \mathbb{N}$. If $S \subseteq \mathbb{R}^2$ is a convex set such that $y \geq 0$ for every $(x, y) \in S$ and $|xy' - yx'| \leq k$ for all $(x, y), (x', y') \in S$, then the area of $S$ is at most $\frac{\pi k}{2}$.*

# A little calculus

Rescalining: we assume $k = 1, y \in [0, 1]$, and define

$$f^{-}(y) = \min\{x : (x, y) \in S\}, f^{+}(y) = \max\{x : (x, y) \in S\}, y' = \sqrt{1 - y^2}$$

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$$\int_0^1 f^+(y)\mathrm{d}y - \int_0^1 \frac{y}{\sqrt{1-y^2}} f^-\left(\sqrt{1-y^2}\right)\mathrm{d}y \le \int_0^1 \frac{1}{\sqrt{1-y^2}}\mathrm{d}y = \frac{\pi}{2}. \tag{1}$$

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by substituting, we get

$$\int_0^1 \frac{y}{\sqrt{1-y^2}} f^-\left(\sqrt{1-y^2}\right)\mathrm{d}y = -1\int_1^0 f^-(y)\mathrm{d}y = \int_0^1 f^-(y)\mathrm{d}y. \tag{2}$$

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We combine (1) and (2) to conclude that the area of $S$ is

$$\int_0^1 f^+(y) - f^-(y)\mathrm{d}y = \int_0^1 f^+(y)\mathrm{d}y - \int_0^1 \frac{y}{\sqrt{1-y^2}} f^-\left(\sqrt{1-y^2}\right)\mathrm{d}y \leq \frac{\pi}{2}.$$

# Upper bound on the size of a $k$-nice set of height $h$

For $\ell \in \mathbb{N}$, we define

$$\rho_\ell = \prod_{\text{primes } p,\, p|\ell} \left(1 - \frac{1}{p}\right);$$

and

$$\alpha_\ell = \max_{1 \le a \le b \le 2\ell} |\{z, a \le z \le b \text{ and } \gcd(z, \ell) = 1\}| - \rho_\ell(b - a + 1);$$

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If $X$ is a set of $n$ consecutive integers, then at most $\rho_\ell n + \alpha_\ell$ elements of $X$ are coprime with $\ell$.

# Size of a $k$-nice set of height $h$ - LP

Consider the following linear program ($\mathrm{LP}_\ell$) with $2\ell$ variables $\sigma_1, \ldots, \sigma_\ell$ and $\tau_1, \ldots, \tau_\ell$:

$$\begin{aligned} \text{maximize} \quad & \sum_{i=1}^{\ell} \rho_i(\tau_i - \sigma_i) & \\ \text{subject to} \quad & \tau_i \geq \sigma_i \geq 0 & \text{for all } 1 \leq i \leq \ell, \text{ and} \\ & -1 \leq i\tau_j - j\sigma_i \leq 1 & \text{for all } 1 \leq i, j \leq \ell. \end{aligned}$$

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- The objective value of the program ($\mathrm{LP}_\ell$) is denoted by $\gamma_\ell$.

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Consider the following linear program (LP$_\ell$) with $2\ell$ variables $\sigma_1, \ldots, \sigma_\ell$ and $\tau_1, \ldots, \tau_\ell$:

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- The objective value of the program (LP$_\ell$) is denoted by $\gamma_\ell$.
- Define $\beta_0 = 1$ and $\beta_\ell = \beta_{\ell-1} + \alpha_\ell + \rho_\ell$.

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**Lemma**

*$\gamma_\ell < 1$ for every $\ell \geq 4$.*

# Size of a $k$-nice set of height $h$ - LP 2

**Lemma**

*Let $k \in \mathbb{N}$. For every $h \in \{1, \ldots, k\}$, every $k$-nice set $Q \subseteq \mathbb{Z}^2$ with height exactly $h$ has at most $\gamma_h k + \beta_h$ elements.*

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| $\ell$ | $\rho_\ell$ | $\alpha_\ell$ | $\gamma_\ell$ | $\beta_\ell$ |
|---|---|---|---|---|
| 1 | 1.0000 | 0.0000 | 1.0000 | 2.0000 |
| 2 | 0.5000 | 0.5000 | 1.0000 | 3.0000 |
| 3 | 0.6667 | 0.6667 | 1.0000 | 4.3333 |
| 4 | 0.5000 | 0.5000 | 0.9722 | 5.3333 |
| 16 | 0.5000 | 0.5000 | 0.9553 | 23.4929 |
| 17 | 0.9412 | 0.9412 | 0.9617 | 25.3753 |
| 18 | 0.3333 | 1.0000 | 0.9576 | 26.7086 |
| 19 | 0.9474 | 0.9474 | 0.9634 | 28.6033 |
| 20 | 0.4000 | 1.2000 | 0.9615 | 30.2033 |

# Proof sketch

Fix $k \in \mathbb{N}$ and $h \in \{1, \ldots, k\}$, Let $Q$ be $x, y$-non-negative, $k$-nice set with height exactly $h$.

- $s_i$ and $t_i$ for $i = 1, \ldots, h$ be the minimum and maximum reals such that $(s_i, i)$ and $(t_i, i)$ are in the convex hull of $Q$.

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- $s_i$ and $t_i$ for $i = 1, \ldots, h$ be the minimum and maximum reals such that $(s_i, i)$ and $(t_i, i)$ are in the convex hull of $Q$.
- Observation $\sigma_i = s_i/k$ and $\tau_i = t_i/k$, $i = 1, \ldots, h$, is a feasible solution of (LP$_h$)

# Proof sketch

Fix $k \in \mathbb{N}$ and $h \in \{1, \dots, k\}$, Let $Q$ be $x, y$-non-negative, $k$-nice set with height exactly $h$.

- $s_i$ and $t_i$ for $i = 1, \dots, h$ be the minimum and maximum reals such that $(s_i, i)$ and $(t_i, i)$ are in the convex hull of $Q$.
- Observation $\sigma_i = s_i/k$ and $\tau_i = t_i/k$, $i = 1, \dots, h$, is a feasible solution of (LP$_h$)

this implies

$$\sum_{i=1}^{h} \rho_i(t_i - s_i) = k \sum_{i=1}^{h} \rho_i(\tau_i - \sigma_i) \leq \gamma_h k. \tag{3}$$

# Proof end

For $i \in \{0, \ldots, h\}$, let $Q_i$ be the points contained in $Q$ with their second coordinate equal to $i$. Then

$$|Q_i| \leq \rho_i(t_i - s_i + 1) + \alpha_i = \rho_i(t_i - s_i) + (\alpha_i + \rho_i)$$

Putting everything together, we get

$$|Q| = \sum_{i=0}^{h} |Q_i| = \left( \sum_{i=1}^{h} \rho_i(t_i - s_i) \right) + 1 + \sum_{i=1}^{h} (\alpha_i + \rho_i) \leq \gamma_h k + \beta_h.$$

This concludes the proof of the lemma.

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This concludes the proof of the lemma.

**Lemma**

*Let $k \in \mathbb{N}$. For every $h \in \{1, \ldots, k\}$, every $k$-nice set $Q \subseteq \mathbb{Z}^2$ with height exactly $h$ has at most $\gamma_h k + \beta_h$ elements.*

# Asymptotics for fixed height

Proved with computer assistance:

**Lemma**

*For every $h \geq 41020$ and every $k \geq h$, the maximum size of a $k$-nice set with height $h$ is at most*

$$\frac{3264\pi}{10255} \cdot k + \frac{4946}{3675} \cdot h + 1.$$

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If the height of a $k$-nice set $Q$ is small, then the size of $Q$ is less than $k$ (note that $\frac{3264\pi}{10255} < 1$). Luckily, the height of a Q is sublinear in $k$.

**Lemma**

*For every $k \in \mathbb{N}$, there exists a k-nice set with maximum size that has height at most $\sqrt{2k}$.*

# Asymptotics conclusions

- Since $\gamma_\ell < 1$ for every $\ell \geq 4$, for every $k \geq 3225$, there exists a maximum size $k$-nice set with height at most 3.

Such $k$-nice sets can be directly analyzed.

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**Lemma**

*For every $k \geq 3$, the maximum size of a $k$-nice set of height at most 3 is*

- $k + 4$ *if* $k \bmod 6 = 2$,
- $k + 3$ *if* $k \bmod 6 \in \{1, 3, 5\}$, *and*
- $k + 2$ *otherwise.*

# Final steps

We are left with determining the sizes of $k$-nice sets for $k \leq 3224$.
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**Lemma**

*For every $k \in \{2, \ldots, 3224\}$, there exists a $k$-nice set with maximum size that has height at most $\sqrt{4k/3}$.*

Thus for $k \geq 1892$, there exists a maximum size $k$-nice set with height at most 3.

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**Lemma**

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Thus for $k \geq 1892$, there exists a maximum size $k$-nice set with height at most 3.

- $k \in \{3, \ldots, 1891\}$: A recursive program based on extra structure gives the proof of the main result.

# A look back at the main result

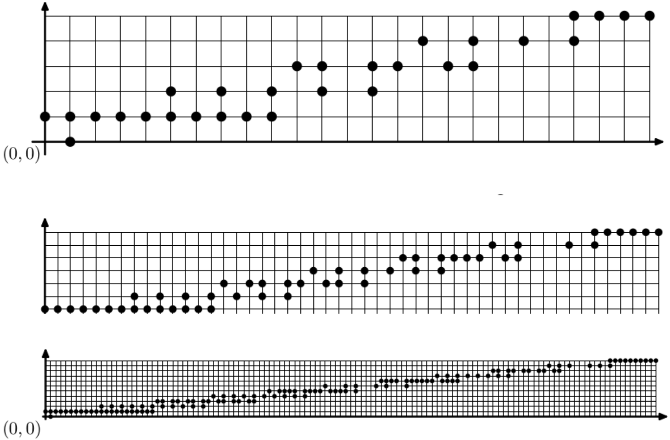
**Theorem**

*For every $k \in \mathbb{N} \setminus K_0$, it holds that*

$$N(\mathbb{T}^2, k) = \begin{cases} k+4 & \textit{if } k \bmod 6 = 2, \\ k+3 & \textit{if } k \bmod 6 \in \{1,3,5\}, \textit{and} \\ k+2 & \textit{otherwise.} \end{cases}$$

*$K_0$ is a special set containing 59 integers.*

# Pictures!



Thank You for Your Attention!