Extension Complexity
Limitations of Linear Programming

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SO WHAT'S THIS ALL ABOUT, THEN?

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Thm : [FPMTW '12]

$$\boxed{xc(\mathrm{CUT}(K_n)) \geq 2^{\Omega(n)}}$$

SO WHAT'S THIS ALL ABOUT, THEN?

STRUCTURE OF THE TALK

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Goals of this talk:

SO WHAT'S THIS ALL ABOUT, THEN?

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Goals of this talk:

- What does it say?
- Why is it true?

SO WHAT'S THIS ALL ABOUT, THEN?

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Thm: [FPMTW '12]

$$\boxed{xc(CUT(K_n)) \geq 2^{\Omega(n)}}$$

Goals of this talk:

- What does it say?
- Why is it true?
- Why is it interesting?

Parsing the result

Parsing the result

Linear Programming

$$
\begin{aligned}
\max \quad & c^T x \\
\text{s.t} \quad & a_1^T x \leq b_1 \\
& a_2^T x \leq b_2 \\
& \vdots \\
& a_m^T x \leq b_m \\
c, a_i & \in \mathbb{R}^n \\
b_i & \in \mathbb{R}
\end{aligned}
$$

PARSING THE RESULT

LINEAR PROGRAMMING

$$\max \; c^T x \quad \longleftarrow \text{linear function}$$

$$\left.\begin{aligned} \text{s.t} \quad a_1^T x &\le b_1 \\ a_2^T x &\le b_2 \\ &\vdots \\ a_m^T x &\le b_m \end{aligned}\right\} \text{linear inequalities}$$

$$c, a_i \in \mathbb{R}^n$$

$$b_i \in \mathbb{R}$$

Parsing the result

Linear Programming

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\begin{aligned}
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$$

- Incredibly versatile!

Parsing the result
Linear Programming

$$\max \; c^T x$$

$$\text{s.t} \quad a_1^T x \le b_1$$
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$$\vdots$$
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$$c, a_i \in \mathbb{R}^n$$
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- Incredibly versatile!
- Efficient both
  - in theory, &
  - in practice.

PARSING THE RESULT

LINEAR PROGRAMMING

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t} \quad & a_1^T x \le b_1 \\ & a_2^T x \le b_2 \\ & \vdots \\ & a_m^T x \le b_m \end{aligned}$$

$$c, a_i \in \mathbb{R}^n$$

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- Incredibly versatile!
- Efficient both
  - in theory, &
  - in practice.
- P-complete.

Parsing the result

Why do we care about Linear Programming?

$\boxed{\text{Linear Programming is P-complete.}}$

PARSING THE RESULT

WHY DO WE CARE ABOUT LINEAR PROGRAMMING?

$\boxed{\text{Linear Programming is P-complete.}}$

Any polynomial-time solvable problem can be converted using logarithmic space into a Linear Programming problem.

PARSING THE RESULT

WHY DO WE CARE ABOUT LINEAR PROGRAMMING?

$\boxed{\text{Linear Programming is P-complete.}}$

Any polynomial-time solvable problem can be converted using logarithmic space into a Linear Programming problem.

Also, Linear programming is solvable in polynomial time.

PARSING THE RESULT

POLYTOPES

PARSING THE RESULT

POLYTOPES

The feasible region of an LP is called a polyhedron.

Parsing the result

Polytopes

The feasible region of an LP is called a polyhedron.

$$\boxed{\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \leq b \end{array}}$$

Linear Program

PARSING THE RESULT

POLYTOPES

The feasible region of an LP is called a polyhedron.

$$\boxed{\begin{array}{ll} \max & c^T x \\ \text{s.t.} & Ax \le b \end{array}} \longrightarrow \boxed{\{x \in \mathbb{R}^n \mid Ax \le b\}}$$

Linear Program

polyhedron

Parsing the result

Polytopes

The feasible region of an LP is called a polyhedron.

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Linear Program

polyhedron

polytope (bounded)

PARSING THE RESULT

POLYTOPES

Minkowski-Weyl theorem

Polytope: bounded intersection of finitely many linear inequalities

PARSING THE RESULT

POLYTOPES

Minkowski - Weyl theorem

Polytope : convex hull of finitely many points

PARSING THE RESULT

POLYTOPES

Minkowski-Weyl theorem

Polytope: bounded intersection of finitely many linear inequalities

($\mathcal{H}$-polytope)

Polytope: convex hull of finitely many points

($\mathcal{V}$-polytope)

PARSING THE RESULT

POLYTOPES

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$\cong$

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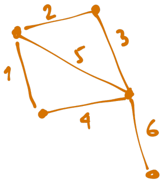
Parsing the result

Polytopes: An example

The CUT Polytope

A graph $G = (V, E)$



Parsing the result

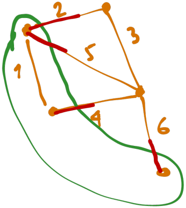
Polytopes: An example

The CUT Polytope

A graph $G = (V, E)$

A cut in G

$\{2, 4, 5, 6\}$



PARSING THE RESULT

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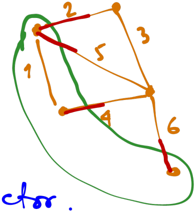

written as a binary vector.

$(0, 1, 0, 1, 1, 1)$

PARSING THE RESULT

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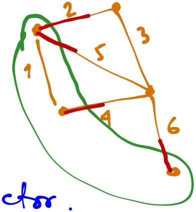

$\{2, 4, 5, 6\}$

written as a binary vector.

$(0, 1, 0, 1, 1, 1)$

Think of cuts as vectors.

Parsing the result

Polytopes: An example

The CUT Polytope

$CUT(K_n)$:= convex hull of all cuts in K_n

PARSING THE RESULT

POLYTOPES: AN EXAMPLE

The CUT Polytope

$\mathrm{CUT}(K_n)$:= convex hull of all cuts in K_n

$\mathrm{CUT}(K_3)$

PARSING THE RESULT

POLYTOPES: AN EXAMPLE

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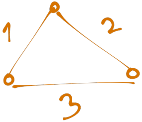

$\mathrm{CUT}(K_3)$

Parsing the result

Polytopes: An example

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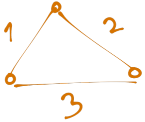

$$CUT(K_3)$$

$$\left\{ \begin{array}{l} (0,0,0), \\ (1,1,0), \\ (1,0,1), \\ (0,1,1) \end{array} \right\}$$

Parsing the result

Polytopes: An example

The CUT Polytope

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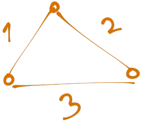

$\mathrm{CUT}(K_3)$

$$\mathrm{conv}\left(\left\{\begin{matrix}(0,0,0), \\ (1,1,0), \\ (1,0,1), \\ (0,1,1)\end{matrix}\right\}\right)$$

PARSING THE RESULT

POLYTOPES: MORE EXAMPLES

Some Polytopes

$\mathrm{CUT}(K_n)$:= convex hull of cuts of K_n

$\mathrm{TSP}(K_n)$:= convex hull of Hamiltonian cycles of K_n

$\mathrm{SAT}(\varphi)$:= convex hull of satisfying assignments of formula ϕ.

$\mathrm{MATCH}(K_n)$:= Convex hull of matchings of K_n

PARSING THE RESULT

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PARSING THE RESULT

WHY DO WE CARE ABOUT POLYTOPES?

Find	$\max c^T x$ s.t $x \in$
largest cut in G	$CUT(G)$

PARSING THE RESULT

WHY DO WE CARE ABOUT POLYTOPES?

Find	max $c^T x$ s.t $x \in$
largest cut in G	CUT(G)
largest Hamiltonian cycle in G	TSP(G)
best satisfying assignment for ϕ	SAT(ϕ)
largest matching in G	MATCH(G)

PARSING THE RESULT

SOLVING PROBLEMS VIA LINEAR PROGRAMMING

Linear Program

$$\max \; c^T x$$
$$\text{s.t.} \; Ax \leq b$$

Feasible region

"Polyhedron" /

"polytope" (bounded)

Extended Formulation

$$\max \; c^T x$$
$$\text{s.t.} \; Ex + Fy \leq c$$

MUST HAVE SAME OPTIMUM FOR ALL $c \in \mathbb{R}^n$

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Linear Program

max $c^T x$

s.t. $Ax \le b$

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MUST HAVE SAME OPTIMUM FOR ALL $c \in \mathbb{R}^n$

PARSING THE RESULT

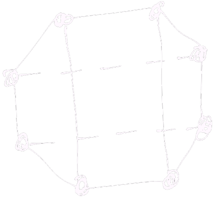
SOLVING PROBLEMS VIA LINEAR PROGRAMMING

$Q = \{(x,y) \mid Ex + Fy \leq c\}$ is an Extended Formulation (EF) of $P = \{x \mid Ax \leq b\}$

iff

$P = \{x \mid \exists y : (x,y) \in Q\}$

i.e. P is a projection of Q



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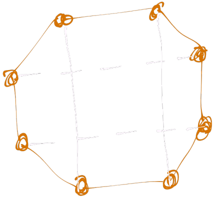

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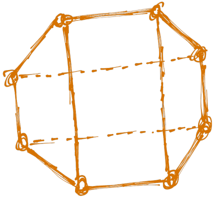

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PARSING THE RESULT

SOLVING PROBLEMS VIA LINEAR PROGRAMMING

size(P) := # inequalities describing P

xc(P) := minimum size of any EF of P

PARSING THE RESULT

FINALLY!

Thm: [FPMTW '12]

$$xc(CUT(K_n)) \geq 2^{\Omega(n)}$$

PARSING THE RESULT

FINALLY!

Thm: [FPMTW '12]

$$xc(CUT(K_n)) \geq 2^{\Omega(n)}$$

Any linear program that is an extended formulation for $\max c^T x$ s.t $x \in CUT(K_n)$ requires $2^{\Omega(n)}$ inequalities.

A narrative..

for what drove the discovery

- Can $xc(P)$ be polynomial
  if linear optimization over P
  NP-hard ?

A NARRATIVE..

FOR WHAT DROVE THE DISCOVERY

- Can xc(P) be polynomial if linear optimization over P NP-hard?

- Is xc(TSP(K_n)) polynomial?

SO WHY IS $\mathrm{xc}(\mathrm{CUT}(K_n))$ HIGH?

NON-NEGATIVE RANK

So why is xc(CUT(K_n)) high?

Non-negative rank

Matrix $M_{m \times n}$

$$\operatorname{rank}(M) := \min_{r} \; M = \sum_{i=1}^{r} T^{i}_{m \times 1} \cdot U^{i}_{1 \times n}$$

SO WHY IS xc(CUT(K_n)) HIGH?

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$$\begin{bmatrix} & \\ & \end{bmatrix}$$

M

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$$\Big[\qquad\Big] = \Big[\,\Big]\cdot[\qquad] + \cdots + \Big[\,\Big]\cdot[\qquad]$$

M

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$$\underbrace{\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}}_{M} = \underbrace{\begin{bmatrix} \, \\ \, \end{bmatrix}\cdot\begin{bmatrix} \quad & \quad \end{bmatrix} + \cdots + \begin{bmatrix} \, \\ \, \end{bmatrix}\cdot\begin{bmatrix} \quad & \quad \end{bmatrix}}_{\#\text{terms} \,\geq\, \operatorname{rank}(M)}$$

SO WHY IS $xc(CUT(K_n))$ HIGH?

NON-NEGATIVE RANK

Matrix $M_{m \times n} \geq 0$

$$\text{rank}_+(M) := \min_{r} \; M = \sum_{i=1}^{r} T^i_{m \times 1} \cdot U^i_{1 \times n}, \quad T^i, U^i \geq 0$$

$$\underset{M}{\begin{bmatrix} \geq 0 \end{bmatrix}} = \underbrace{\begin{bmatrix} \geq 0 \end{bmatrix} \cdot \begin{bmatrix} \geq 0 \end{bmatrix} + \cdots + \begin{bmatrix} \geq 0 \end{bmatrix} \cdot \begin{bmatrix} \geq 0 \end{bmatrix}}_{\#\text{terms} \geq \text{rank}_+(M)}$$

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$$T^i, U^i \geq 0$$

Theorem (Yannakakis '89)

Let P be a polytope. Then,

$$\mathrm{xc}(P) = \mathrm{rank}_+(S(P))$$

So why is $\mathrm{xc}(\mathrm{CUT}(K_n))$ high?

The Slack Matrix

Let P be a polytope.

$$P = \left\{ x \;\middle|\; \begin{array}{l} a_1^T x \le b_1 \\ a_2^T x \le b_2 \\ \vdots \\ a_m^T x \le b_m \end{array} \right\} = \mathrm{conv}\left(\{v_1, v_2, \cdots, v_n\}\right)$$

SO WHY IS xc(CUT(K_n)) HIGH?

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Slack matrix $S(P)$

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$$S_{ij} = b_i - a_i^T v_j$$

SO WHY IS $\mathrm{xc}(\mathrm{CUT}(K_n))$ HIGH?

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Slack matrix $S(P)$

$$S_{ij} = b_i - a_i^T v_j \geq 0$$

SO WHY IS xc(CUT(K_n)) HIGH?

THE SLACK MATRIX

Matrix $M_{m \times n} \geq 0$

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RECALL

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RECALL

Theorem (FPMTW '12)

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SO WHY IS xc(CUT(K_n)) HIGH?

RECTANGLE COVERING BOUND

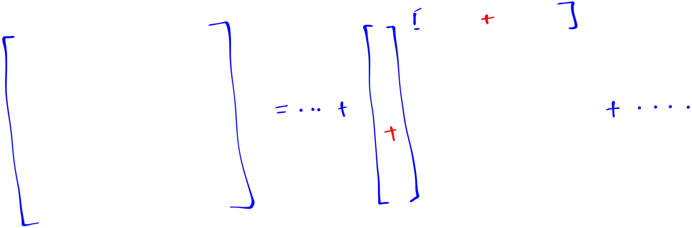
SO WHY IS xc(CUT(K_n)) HIGH?

RECTANGLE COVERING BOUND

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \dots + \begin{bmatrix} \\ \\ \end{bmatrix}^{!}\;] + \dots$$

So why is xc(CUT(K_n)) high?

Rectangle Covering Bound



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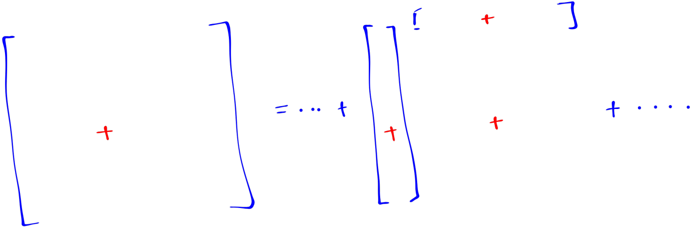

So why is xc(CUT(K_n)) high?

Rectangle Covering Bound

$$\left[\qquad\right] = \dots + \underbrace{\left[\,\right]\left[\qquad\right]}_{i\text{-th term}} + \dots$$

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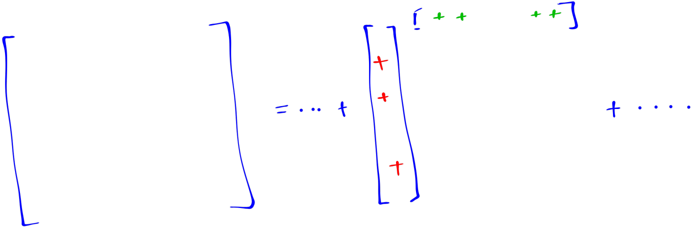

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RECTANGLE COVERING BOUND

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} = \dots + \begin{bmatrix} + \\ + \\ \\ + \end{bmatrix} \begin{bmatrix} + & + & & + & + \end{bmatrix} + \dots$$

R_i := rows with positive entries

C_i := columns with positive entries

SO WHY IS $\mathrm{xc}(\mathrm{CUT}(K_n))$ HIGH?

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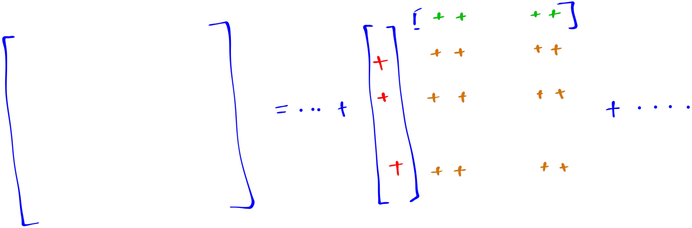

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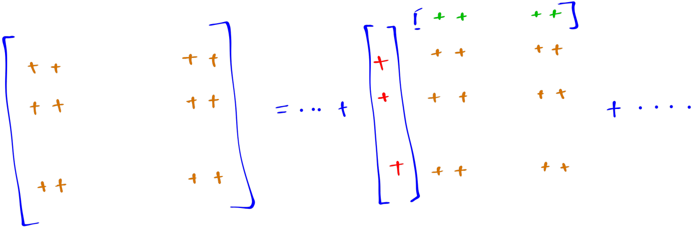

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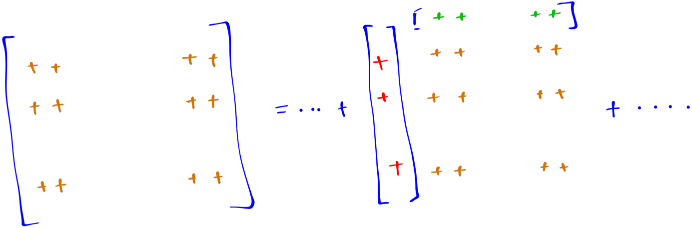


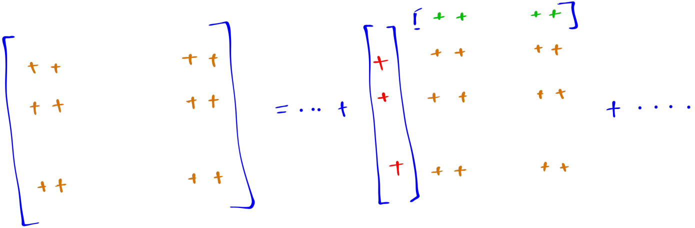
$R_i :=$ rows with positive entries

$C_i :=$ columns with positive entries

$$\boxed{M_{k,l} > 0 \text{ iff } \exists i : k \in R_i,\ l \in C_i}$$

SO WHY IS $\mathrm{xc}(\mathrm{CUT}(K_n))$ HIGH?

RECTANGLE COVERING BOUND



R_i, C_i 1-rectangle

$\mathrm{rank}_+(M) = r \Rightarrow$ can cover positive entries of M with r 1-rectangles.

SO WHY IS $\mathrm{xc}(\mathrm{CUT}(K_n))$ HIGH?

A HARD MATRIX

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A HARD MATRIX

M row indices $a \in \{0,1\}^n$

col indices $b \in \{0,1\}^n$

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A HARD MATRIX

M

row indices $a \in \{0,1\}^n$

col indices $b \in \{0,1\}^n$

$$M_{a,b} = (a^T b - 1)^2$$

SO WHY IS xc(CUT(K_n)) HIGH?

A HARD MATRIX

M row indices $a \in \{0,1\}^n$

col indices $b \in \{0,1\}^n$

$$M_{a,b} = (a^T b - 1)^2$$

- Need atleast $2^{n/2}$ 1-rectangles to cover positive entries of M.

SO WHY IS $\mathrm{xc}(\mathrm{CUT}(K_n))$ HIGH?

A HARD MATRIX HIDES IN $\mathcal{S}(\mathrm{CUT}(K_n))$

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- M is a sub-matrix of $\mathcal{S}(\mathrm{CUT}(K_n))$

$$\Rightarrow \boxed{xc((\mathrm{CUT}(K_n))) \geq 2^{n/2}}$$

And why...

is the extension complexity high other times?

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Yields lower bounds for

- TSP
- SAT
- INDEPENDENT SET
- ...

AND WHY...

IS THE EXTENSION COMPLEXITY HIGH OTHER TIMES?

- A finer argument called "Hyperplane separtion lower bound" shows that

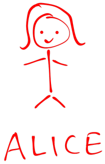
$$xc(MATCH(K_n)) \geq 2^{\Omega(n)}$$

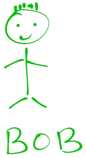
  - [Rothvoß '14]

Okay, so?

Okay, so?

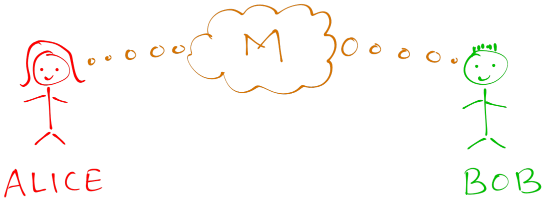
Communication Complexity in Expectation





OKAY, SO?

COMMUNICATION COMPLEXITY IN EXPECTATION



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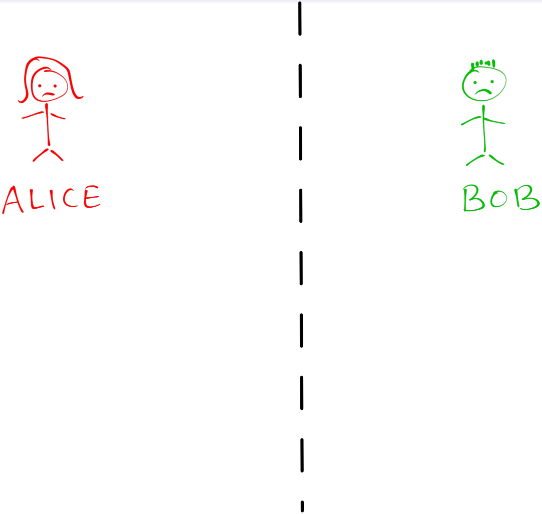
Communication Complexity in Expectation



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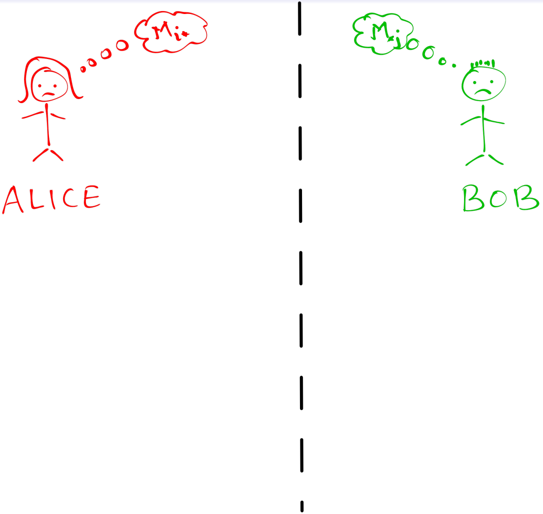
Communication Complexity in Expectation



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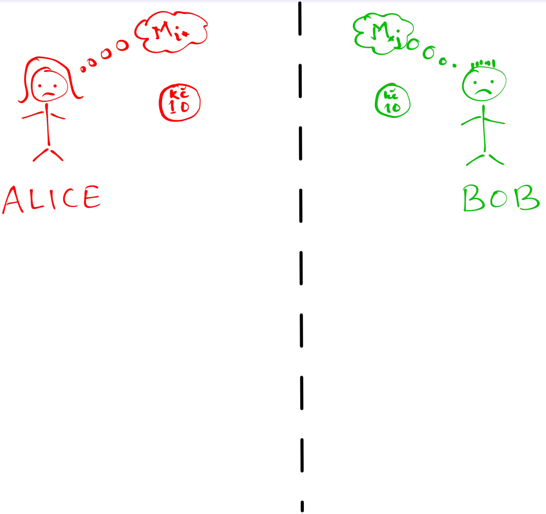
Communication Complexity in Expectation



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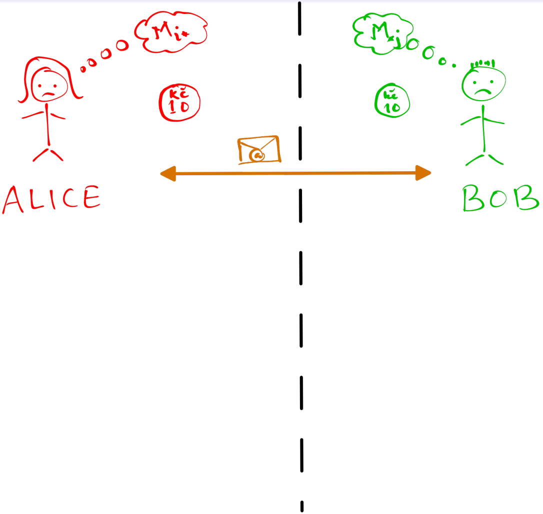

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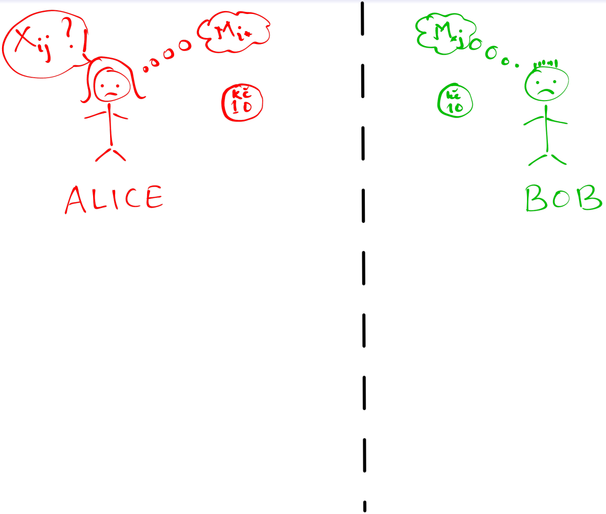

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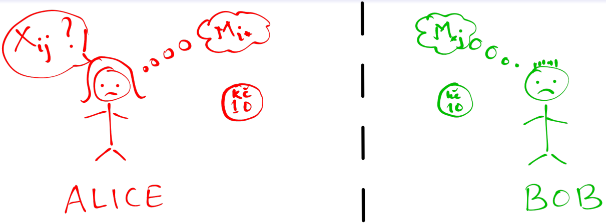
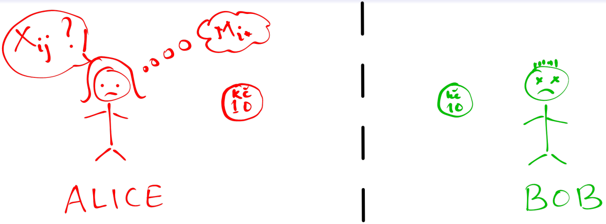


If $\mathbb{E}[X_{ij}] \neq M_{ij}$

Okay, so?

Communication Complexity in Expectation



If $\mathbb{E}[X_{ij}] \neq M_{ij}$

OKAY, SO?

CONNECTION TO EXTENDED FORMULATIONS

Cost of saving Bob $(ecc(M))$

$= \max_{ij}$ size of message to ensure $\mathbb{E}[X_{ij}] = M_{ij}$

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CONNECTION TO EXTENDED FORMULATIONS

Cost of saving Bob (ecc(M))

$= \max_{ij}$ size of message to ensure $\mathbb{E}[X_{ij}] = M_{ij}$

$$\boxed{\begin{array}{l} \bullet\ X_{ij} \geq 0 \\ \bullet\ M \geq 0 \end{array}}$$

OKAY, SO?

CONNECTION TO EXTENDED FORMULATIONS

Cost of saving Bob $(\mathrm{ecc}(M))$

$$= \max_{ij} \text{ size of message to ensure } \mathbb{E}[X_{ij}] = M_{ij}$$

Theorem (FFGT '12)

$$\boxed{xc(P) \approx 2^{\mathrm{ecc}(S(P))}}$$

A VERY GENERAL PERSPECTIVE

GENERALIZED COMMUNICATION COMPLEXITY

- What's a message?
  - sequence of bits (classical)
  - sequence of qubits (quantum)
  - sequence of more exotic objects

A very general perspective

Generalized communication complexity

- What's a message?
  - sequence of bits (classical)
  - sequence of qubits (quantum)
  - sequence of more exotic objects

A VERY GENERAL PERSPECTIVE

CONVEX CONE PROGRAMMING

- How to solve a linear program?
  - Linear extensions
  - Semidefinite extensions
  - General conic extensions

$$\max \; c^T x$$
$$\text{s.t.} \; Ax \leq b$$

A very general perspective

Convex cone programming

- How to solve a linear program?
  - Linear extensions
  - Semidefinite extensions
  - General conic extensions

$$\max c^T x \quad \text{s.t} \quad Ax \leq b$$

$$\max c^T x \quad \text{s.t} \quad Ex + Fy = g, \quad y \geq 0$$

A very general perspective

Convex cone programming

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  - Semidefinite extensions
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$$\max \; c^T x \quad \text{s.t} \quad Ax \le b$$

$$\max \; c^T x \quad \text{s.t} \quad Ex + Fy = g, \quad y \in \mathbb{R}^{r}_{\ge 0}$$

Non-negative orthant

A VERY GENERAL PERSPECTIVE

CONVEX CONE PROGRAMMING

- How to solve a linear program?
  - Linear extensions
  - Semidefinite extensions
  - General conic extensions

$$\max \; c^T x \quad \text{s.t} \quad Ax \leq b$$

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Cone of PSD Matrices

A VERY GENERAL PERSPECTIVE

CONVEX CONE PROGRAMMING

- How to solve a linear program?
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$$\max \; c^T x \quad \text{s.t.} \quad Ax \leq b$$

$$\max \; c^T x \quad \text{s.t.} \quad Ex + Fy = g, \quad y \in C$$

A general closed convex cone

A very general perspective

A general equivalence

Theorem : $xc(P) = 2^{ecc(S(P))}$

A VERY GENERAL PERSPECTIVE

A GENERAL EQUIVALENCE

Theorem: $xc(P) = 2^{ecc(S(P))}$

Communication complexity $\approx$ Extension Complexity

"Communication"	"Extension"
- Classical	Linear
- Quantum	Semidefinite
- ???	Convex

Thank you!

Questions? Comments?