On Counting Elements in Product Sets

Alessandro Fazzari

Università di Genova

Algebra Colloquium, Charles University, Prague

October 14, 2025

On Counting Elements in Product Sets

Alessandro Fazzari

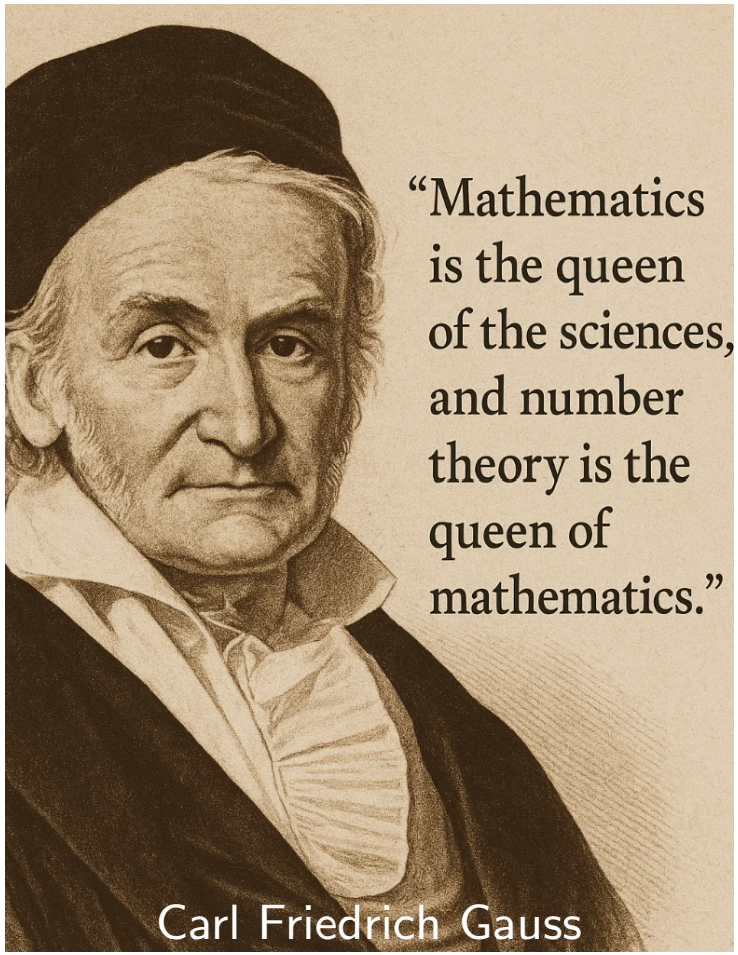
Università di Genova

Algebra Colloquium, Charles University, Prague

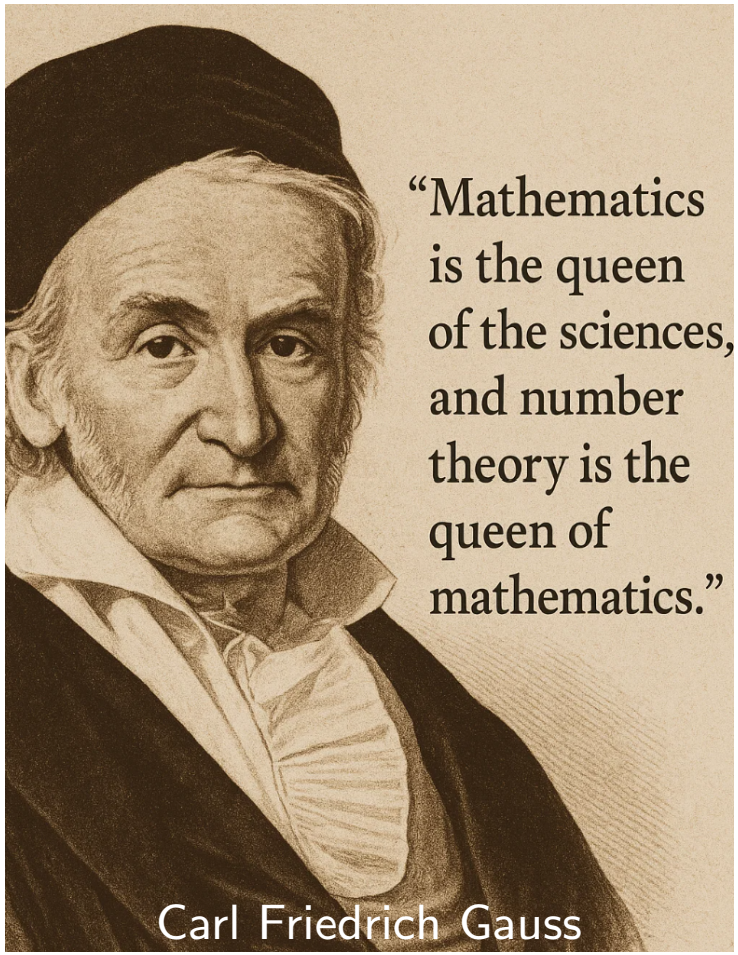
October 14, 2025

(Happy Birthday, Ale!)

Analytic Number Theory and Logarithms



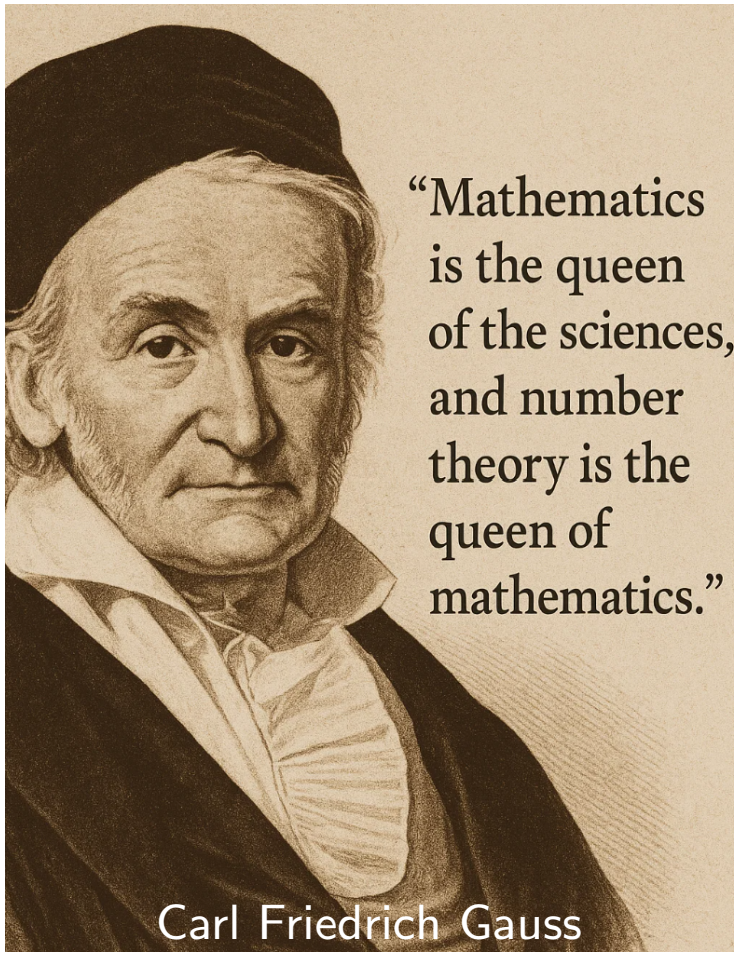
Analytic Number Theory and Logarithms



What is Analytic Number Theory?

Number theory is the branch of mathematics that studies objects of an arithmetic nature; the prototypical examples being prime numbers. In particular, analytic number theory employs tools from complex analysis to investigate arithmetic questions, for instance using the residue theorem.

Analytic Number Theory and Logarithms



What is Analytic Number Theory?

Number theory is the branch of mathematics that studies objects of an arithmetic nature; the prototypical examples being prime numbers. In particular, analytic number theory employs tools from complex analysis to investigate arithmetic questions, for instance using the residue theorem.

What is Analytic Number Theory known for?

Lots of logs, loglogs, logloglogs . . .

Analytic Number Theory and Logarithms

Analytic Number Theory and Logarithms

Analytic Number Theory and Logarithms

But are these logarithms truly necessary, or do analytic number theorists just have a perverse sense of aesthetics?

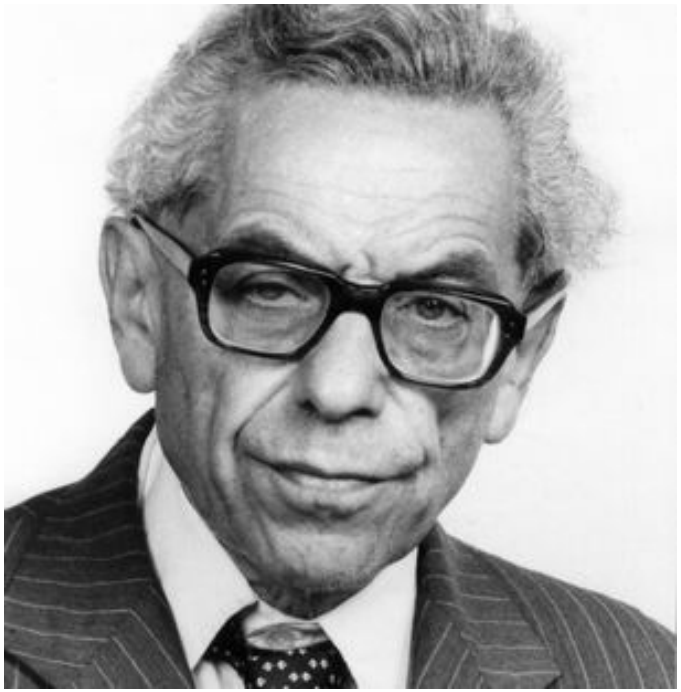
Many mathematicians claim that these iterated logarithms naturally appear in nature.

Analytic Number Theory and Logarithms

But are these logarithms truly necessary, or do analytic number theorists just have a perverse sense of aesthetics?

Many mathematicians claim that these iterated logarithms naturally appear in nature.

Let's see an example: the multiplication table problem, posed by Erdős in 1955.



The Multiplication Table Problem

Back to elementary school–let's look at the 10×10 multiplication table:

$\times$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

The Multiplication Table Problem

Back to elementary school–let's look at the 10×10 multiplication table:

$\times$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

There are 100 entries in the above table; how many distinct integers appear in the table?

The Multiplication Table Problem

Back to elementary school–let's look at the 10×10 multiplication table:

$\times$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

There are 100 entries in the above table; how many distinct integers appear in the table? There are 42 numbers appearing in the 10×10 table.

The Multiplication Table Problem

Given $N \in \mathbb{N}$, let $\mathcal{M}(N)$ denote the number of distinct products ab, where $1 \leq a, b \leq N$. These are precisely the entries in an $N \times N$ multiplication table.

The Multiplication Table Problem

Given $N \in \mathbb{N}$, let $\mathcal{M}(N)$ denote the number of distinct products ab, where $1 \leq a, b \leq N$. These are precisely the entries in an $N \times N$ multiplication table.

How many distinct products occur, i.e., what is $\mathcal{M}(N)$?

The Multiplication Table Problem

Given $N \in \mathbb{N}$, let $\mathcal{M}(N)$ denote the number of distinct products ab, where $1 \leq a, b \leq N$. These are precisely the entries in an $N \times N$ multiplication table.

How many distinct products occur, i.e., what is $\mathcal{M}(N)$?

- The table has N^2 entries, i.e., $\mathcal{M}(N) \leq N^2$
- It is symmetric, so we have repetitions: $\mathcal{M}(N) \leq \frac{N(N+1)}{2}$.
- How many distinct entries does it have?
- In the previous slide, we saw that $\mathcal{M}(10) = 42$.
- Other examples can be computed:

N	5	10	20	40	80	160	320	640	1000
$\mathcal{M}(N)$	14	42	109	321	784	1792	3968	8704	15864

The Multiplication Table Problem

What would you conjecture about $\mathcal{M}(N)$ asymptotically, i.e. as $N \to \infty$?

The Multiplication Table Problem

What would you conjecture about $\mathcal{M}(N)$ asymptotically, i.e. as $N \to \infty$?

It is natural to compare the size of $\mathcal{M}(N)$ with the total number of entries N^2:

N	$\frac{\mathcal{M}(N)}{N^2}$
5	0.5600
10	0.4200
20	0.3800
40	0.3231
80	0.3030
160	0.2802
320	0.2671
640	0.2538
1000	0.2481
2000	0.2399
8000	0.2267
16000	0.2215
32000	0.2166

The Multiplication Table Problem

So perhaps $\dfrac{\mathcal{M}(N)}{N^2} \to 0$? For friends, $\mathcal{M}(N) = o(N^2)$?

And, if so, how fast does that ratio go to 0? Explicit bounds on the order of magnitude of $\mathcal{M}(N)$? Even asymptotic formulae?

The Multiplication Table Problem

So perhaps $\dfrac{\mathcal{M}(N)}{N^2} \to 0$? For friends, $\mathcal{M}(N) = o(N^2)$?

And, if so, how fast does that ratio go to 0? Explicit bounds on the order of magnitude of $\mathcal{M}(N)$? Even asymptotic formulae?

- Is $\mathcal{M}(N)$ of the form N^{2-c_1}?
- Or is $\mathcal{M}(N)$ of the form $N^2/(\log N)^{c_2}$?
- Or how about $\mathcal{M}(N)$ of the form $N^2/\exp((\log N)^{c_3})$?

The Multiplication Table Problem

So perhaps $\dfrac{\mathcal{M}(N)}{N^2} \to 0$? For friends, $\mathcal{M}(N) = o(N^2)$?

And, if so, how fast does that ratio go to 0? Explicit bounds on the order of magnitude of $\mathcal{M}(N)$? Even asymptotic formulae?

- Is $\mathcal{M}(N)$ of the form N^{2-c_1}?
- Or is $\mathcal{M}(N)$ of the form $N^2/(\log N)^{c_2}$?
- Or how about $\mathcal{M}(N)$ of the form $N^2/\exp((\log N)^{c_3})$?

N	$\frac{\mathcal{M}(N)}{N^2}$	c_1	c_2	c_3
5	0.5600	0.3603	1.2184	1.1453
10	0.4200	0.3768	1.0401	0.1704
20	0.3800	0.3230	0.8819	0.0300
40	0.3231	0.3063	0.8655	0.0935
80	0.3030	0.2725	0.8081	0.1200
160	0.2802	0.2507	0.7832	0.1482
320	0.2671	0.2289	0.7533	0.1585
640	0.2538	0.2122	0.7349	0.1692
1000	0.2481	0.2018	0.7213	0.1718

Acknowledgment:
Table adapted from work of Pomerance & Kurlberg & Lagarias.

The Function of Prime Divisors

Besides numerics, a key tool to understand the multiplication table problem is the prime divisor function Ω of an integer n:

$$\begin{aligned}\Omega(n) :&= \text{number of prime divisors of } n, \text{counted with multiplicity} \\ &= \sum_{p^\alpha || n} \alpha = \alpha_1 + \cdots + \alpha_k, \quad \text{if } n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}.\end{aligned}$$

The Function of Prime Divisors

Besides numerics, a key tool to understand the multiplication table problem is the prime divisor function Ω of an integer n:

$$\begin{aligned}\Omega(n) :&= \text{number of prime divisors of } n, \text{counted with multiplicity} \\ &= \sum_{p^\alpha || n} \alpha = \alpha_1 + \cdots + \alpha_k, \quad \text{if } n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}.\end{aligned}$$

Note that the additive function $\Omega(n)$ is pretty nasty; its behavior is very erratic:

- If $n = p$ is prime, then $\Omega(n) = 1$;
- If $n = 2^k$, then $\Omega(n) = k$.

In particular, for any $n \geq 2$,

$$1 \leq \Omega(n) \ll \log n$$

with both inequalities being optimal.

The Function of Prime Divisors

The average behavior of Ω is much more regular. One can show that

$$\frac{1}{N}\sum_{n\leq N}\Omega(n)\sim \log\log N,$$

The Function of Prime Divisors

The average behavior of Ω is much more regular. One can show that

$$\frac{1}{N}\sum_{n\le N}\Omega(n)\sim \log\log N,$$

Indeed,

$$\begin{aligned}\frac{1}{N}\sum_{n\le N}\Omega(n)\approx \frac{1}{N}\sum_{n\le N}\sum_{p\mid n}1=\frac{1}{N}\sum_{p\le N}\sum_{\substack{n\le N\\ n\equiv 0\ (p)}}1=\frac{1}{N}\sum_{p\le N}\sum_{m\le N/p}1\\ =\frac{1}{N}\sum_{p\le N}\left(\frac{N}{p}+O(1)\right)=\sum_{p\le N}\frac{1}{p}+O\left(\frac{1}{N}\sum_{p\le N}1\right)\\ =\log\log N+O(1).\end{aligned}$$

The Function of Prime Divisors

With the same technique, one sees that

$$\frac{1}{N}\sum_{n\leq N}\Omega(n)^2 = (\log\log N)^2 + O(\log\log N).$$

The Function of Prime Divisors

With the same technique, one sees that

$$\frac{1}{N}\sum_{n\leq N}\Omega(n)^2 = (\log\log N)^2 + O(\log\log N).$$

Via Chebyshev's inequality, these two computations imply that

$$\Omega(n) \sim \log\log n \qquad \text{“typically”}.$$

In other words, this means that not only does $\Omega(n)$ have average value $\log\log n$, but this is also its normal order. More precisely, $\Omega(n) \sim \log\log n$ for all $n \leq N$ outside an exceptional set of size $o(N)$.

Erdős' idea

Let's now prove that

$$\mathcal{M}(N) = o(N^2).$$

Recall that for $\sqrt{N} < n \leq N$, we have $\Omega(n) \sim \log\log N$, typically. Most products ab have both $a > \sqrt{N}$ and $b > \sqrt{N}$. Therefore:

- On one hand, $\sqrt{N} < a, b \leq N$ implies that $N < ab \leq N^2$ and for most such products, $\Omega(ab) \sim \log\log N^2 \sim \log\log N$;
- On the other hand, the complete additivity of Ω implies that $\Omega(ab) = \Omega(a) + \Omega(b) \sim 2\log\log N$.

Erdős' idea

Let's now prove that

$$\mathcal{M}(N) = o(N^2).$$

Recall that for $\sqrt{N} < n \leq N$, we have $\Omega(n) \sim \log\log N$, typically. Most products ab have both $a > \sqrt{N}$ and $b > \sqrt{N}$. Therefore:

- On one hand, $\sqrt{N} < a, b \leq N$ implies that $N < ab \leq N^2$ and for most such products, $\Omega(ab) \sim \log\log N^2 \sim \log\log N$;
- On the other hand, the complete additivity of Ω implies that $\Omega(ab) = \Omega(a) + \Omega(b) \sim 2\log\log N$.

This observation (made by Erdős) suggests that the products ab are not typical integers, as they have an unusually large number of prime divisors.

Consequently, this immediately implies that, at least, $\mathcal{M}(N) = o(N^2)$.

The Function of Prime Divisors, refined

The distribution of prime divisors can be studied much more precisely. The famous Erdős-Kac theorem states that $\Omega(n)$ converges in distribution to a Gaussian. Namely, for any fixed $V > 0$,

$$\frac{1}{N}\#\left\{n \le N : \frac{\Omega(n) - \log\log N}{\sqrt{\log\log N}} > V\right\} \sim \int_V^\infty e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}.$$

In other words, most integers have $\Omega(n)$ close to $\log\log n$, and the typical deviations of order $\sqrt{\log\log n}$ are Gaussian.

The Function of Prime Divisors, refined

The distribution of prime divisors can be studied much more precisely. The famous Erdős-Kac theorem states that $\Omega(n)$ converges in distribution to a Gaussian. Namely, for any fixed $V > 0$,

$$\frac{1}{N}\#\left\{n \le N : \frac{\Omega(n) - \log\log N}{\sqrt{\log\log N}} > V\right\} \sim \int_V^\infty e^{-x^2/2}\frac{dx}{\sqrt{2\pi}}.$$

In other words, most integers have $\Omega(n)$ close to $\log\log n$, and the typical deviations of order $\sqrt{\log\log n}$ are Gaussian.

Moreover, the large deviations of the function Ω exhibit Poisson-like behavior; for any $B > 1$,

$$\frac{1}{N}\#\left\{n \le N : \Omega(n) > B\log\log N\right\} \asymp e^{-\phi(B)\log\log N} = (\log N)^{-\phi(B)},$$

where $\phi(B) = B\log B - B + 1$.

The Answer to the Multiplication Table Problem

Using these more refined results (and many other techniques), Erdős (1960) showed that

$$\mathcal{M}(N) = \frac{N^2}{(\log N)^{\delta+o(1)}}, \quad \text{where } \delta = 1 - \frac{1+\log\log 2}{\log 2} \approx 0.08607.$$

This means that, for any fixed $\varepsilon > 0$,

$$\frac{N^2}{(\log N)^{\delta}}(\log N)^{-\varepsilon} \ll \mathcal{M}(N) \ll \frac{N^2}{(\log N)^{\delta}}(\log N)^{\varepsilon}.$$

The Answer to the Multiplication Table Problem

Using these more refined results (and many other techniques), Erdős (1960) showed that

$$\mathcal{M}(N) = \frac{N^2}{(\log N)^{\delta+o(1)}}, \quad \text{where } \delta = 1 - \frac{1+\log\log 2}{\log 2} \approx 0.08607.$$

This means that, for any fixed $\varepsilon > 0$,

$$\frac{N^2}{(\log N)^{\delta}}(\log N)^{-\varepsilon} \ll \mathcal{M}(N) \ll \frac{N^2}{(\log N)^{\delta}}(\log N)^{\varepsilon}.$$

Building on work of Tenenbaum, Ford (2008) finally determined the order of magnitude of $\mathcal{M}(N)$, proving that

$$\frac{N^2}{(\log N)^{\delta}(\log\log N)^{3/2}} \ll \mathcal{M}(N) \ll \frac{N^2}{(\log N)^{\delta}(\log\log N)^{3/2}}.$$

The Answer to the Multiplication Table Problem

Using these more refined results (and many other techniques), Erdős (1960) showed that

$$\mathcal{M}(N) = \frac{N^2}{(\log N)^{\delta+o(1)}}, \quad \text{where } \delta = 1 - \frac{1+\log\log 2}{\log 2} \approx 0.08607.$$

This means that, for any fixed $\varepsilon > 0$,

$$\frac{N^2}{(\log N)^{\delta}}(\log N)^{-\varepsilon} \ll \mathcal{M}(N) \ll \frac{N^2}{(\log N)^{\delta}}(\log N)^{\varepsilon}.$$

Building on work of Tenenbaum, Ford (2008) finally determined the order of magnitude of $\mathcal{M}(N)$, proving that

$$\frac{N^2}{(\log N)^{\delta}(\log\log N)^{3/2}} \ll \mathcal{M}(N) \ll \frac{N^2}{(\log N)^{\delta}(\log\log N)^{3/2}}.$$

The constant is still unknown, so an asymptotic formula for $\mathcal{M}(N)$ remains out of reach with current technology.

The Answer to the Multiplication Table Problem

The upshot of the multiplication table problem is:

- The multiplication table numbers have an unusually large number of prime divisors, and hence they are not typical integers.
- The multiplication table numbers are typically balanced products ab, say with $a, b > \sqrt{N}$, and each factor "contributes" with $\sim \log\log N$ prime divisors.

On the Asymptotic Density of Product Sets

Let's look at the analogous problem for infinite sets of natural numbers. In this context, the role of the cardinality is played by the natural density; given a set $A \subseteq \mathbb{N}$,

$$d(A) := \lim_{x \to \infty} \frac{\#(A \cap [1, x])}{x}.$$

In 2021, Bettin-Koukoulopoulos-Sanna studied the product set

$$A \cdot A := \{a_1 a_2 : a_1, a_2 \in A\},$$

and proved the following result:

If A is a set with natural density 1, then the set $A \cdot A$ has also density 1.

On the Asymptotic Density of Product Sets

Note that this case differs from the multiplication table problem. The difference lies in the fact that many elements of $A \cdot A$ come from very unbalanced products, meaning products ab such that the sizes of a and b are completely different.

On the Asymptotic Density of Product Sets

Note that this case differs from the multiplication table problem. The difference lies in the fact that many elements of $A \cdot A$ come from very unbalanced products, meaning products ab such that the sizes of a and b are completely different.

Idea: any integer n can be factored as

$$n = n_{\text{smooth}} \cdot n_{\text{rough}},$$

where n_{smooth} and n_{rough} are the products of its "small" and "large" prime factors, respectively. If $n \notin A \cdot A$, then at least one of these factors must be missing from A, meaning either $n_{\text{smooth}} \notin A$ or $n_{\text{rough}} \notin A$. If the product set $A \cdot A$ does not have density 1, then A must lack its expected proportion of either smooth or rough numbers. Consequently, A itself cannot have density 1.

On the Asymptotic Density of Product Sets

More precisely, denoting

$$R_x(A) = 1 - \frac{\#(A \cap [1, x])}{x},$$

Bettin-Koukoulopoulos-Sanna proved that

If

$$R_x(A) \ll (\log x)^{-a} \quad \text{for some } a \in (0, 1)$$

then

$$R_x(A \cdot A) \ll (\log x)^{-\frac{a^2}{1+a}+o(1)}.$$

On the Asymptotic Density of Product Sets

More precisely, denoting

$$R_x(A) = 1 - \frac{\#(A \cap [1, x])}{x},$$

Bettin-Koukoulopoulos-Sanna proved that

If

$$R_x(A) \ll (\log x)^{-a} \text{ for some } a \in (0, 1)$$

then

$$R_x(A \cdot A) \ll (\log x)^{-\frac{a^2}{1+a}+o(1)}.$$

Equivalently, letting

$$\psi(a) := \sup\{b > 0 : R_x(A \cdot A) \ll (\log x)^{-b} \ \forall A \subseteq \mathbb{N} \text{ with } R_x(A) \ll (\log x)^{-a}\},$$

their result can be stated as

$$\psi(a) \geq \frac{a^2}{1+a},$$

for $a \in (0, 1)$.

On the Asymptotic Density of Product Sets

In joint work with Sandro Bettin and Matteo Bordignon, we proved an upper bound for ψ, providing a set of density 1 such that its square is "as small as possible".

Theorem (Bettin, Bordignon, F., 2025)

Letting

$$\psi(a) := \sup\{b > 0 : R_x(A \cdot A) \ll (\log x)^{-b} \ \forall A \subseteq \mathbb{N} \text{ with } R_x(A) \ll (\log x)^{-a}\},$$

we have

$$\psi(a) \leq \begin{cases} a & \text{for } a \in (0, 0.117) \\ 6.51a^2 + o(a^2) & \text{as } a \to 0. \end{cases}$$

On the Asymptotic Density of Product Sets

In joint work with Sandro Bettin and Matteo Bordignon, we proved an upper bound for ψ, providing a set of density 1 such that its square is "as small as possible".

Theorem (Bettin, Bordignon, F., 2025)

Letting

$$\psi(a) := \sup\{b > 0 : R_x(A \cdot A) \ll (\log x)^{-b} \ \forall A \subseteq \mathbb{N} \text{ with } R_x(A) \ll (\log x)^{-a}\},$$

we have

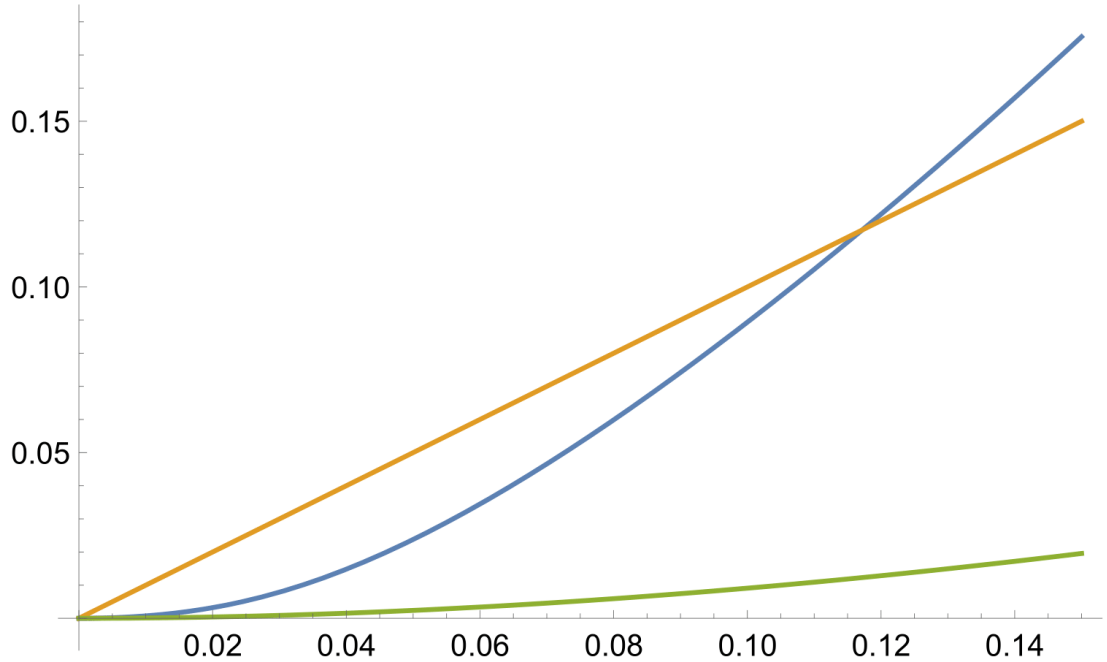
$$\psi(a) \leq \begin{cases} a & \text{for } a \in (0, 0.117) \\ 6.51a^2 + o(a^2) & \text{as } a \to 0. \end{cases}$$

To construct a set A such that $\mathbb{N} \setminus (A \cdot A)$ is large, we define A as the set of integers that have "not too few" large prime divisors. This way, elements of $A \cdot A$ have "too many" prime divisors, as in the multiplication table problem.

On the Asymptotic Density of Product Sets

In particular, our result implies that $\psi(a)$ decays quadratically as $a \to 0^+$:

$$1 \le \liminf_{a \to 0^+} \frac{\psi(a)}{a^2} \le \limsup_{a \to 0^+} \frac{\psi(a)}{a^2} \le \frac{2}{1 - \log 2} = 6.51778\ldots.$$



The functions $\frac{a^2}{1+a}$ (green), $K(a)$ (blue) and a (orange) for $0 \le a \le 0.15$. The function ψ lies between the green curve and the minimum of the blue and the orange curves.

References

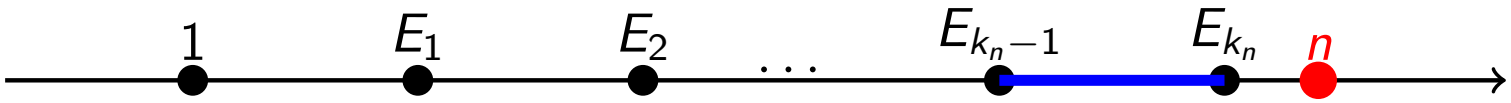
- S. Bettin; M. Bordignon; A. Fazzari, *On products of sets of natural density one*. Preprint (arXiv:2504.01665).
- S. Bettin; D. Koukoulopoulos; C. Sanna, *A note on the natural density of product sets*, Bull. Lond. Math. Soc. 53 (2021), no. 5, 1407–1413.
- P. Erdös, *Some remarks on number theory*, Riveon Lematematika 9 (1955), 45–48.
- P. Erdös, *An asymptotic inequality in the theory of numbers*, Vestnik Leningrad. Univ. 15 (1960), no. 13, 41–49.
- K. Ford, *The distribution of integers with a divisor in a given interval*, Ann. of Math. (2) 168 (2008), no. 2, 367–433.
- P. Kurlberg; J. Lagarias; C. Pomerance, *Product-free sets with high density*. Acta Arith. 155 (2012), no. 2, 163–173.

Sketch of the Proof

For parameters $y > 1$ and $\frac{1}{2} < B < 1$, let's discretize the positive real axis according to the points

$$E_k = e^{e^{y^k}}.$$

Given an integer n, denote by k_n the largest integer k such that $E_k < n$.



Consider the function

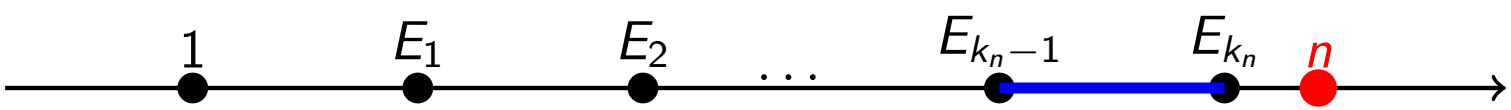
$$\Omega_k(n) := \#\{p \mid n : E_{k-1} < p < E_k, \text{ with multiplicity}\}.$$

Define the set

$$A := \{n \in [E_0, \infty) : \Omega_{k_n}(n) > B(\log\log E_{k_n} - \log\log E_{k_n-1})\}.$$

Note that $\log\log E_k - \log\log E_{k-1}$ is the expected number of prime divisors between E_{k-1} and E_k.

Sketch of the Proof



One sees that

$$R_x(A) \ll (\log x)^{-\phi(B)(\frac{1}{y}-\frac{1}{y^2})}.$$

Now consider $n = ab \in (A \cdot A) \cap [1, x]$. Then:

- if $a, b \in [E_{k_n-1}, E_{k_n}]$, then $a, b \in A$ implies that their product ab has $2B(\log\log E_{k_n} - \log\log E_{k_n-1})$ prime divisors in $[E_{k_n-1}, E_{k_n}]$
  $\Longrightarrow$ unlikely $(2B > 1)$ $\Longrightarrow$ $A \cdot A$ "small" $\Longrightarrow$ lower bound for $R_x(A \cdot A)$.
- if $a \in [E_{k_n-1}, E_{k_n}]$ and $b \in [E_{r-1}, E_r]$ for some $r < k_n$, then the conditions on the number of prime divisors on disjoint intervals are independent. Using classical large deviation results, one gets a lower bound for $R_x(A \cdot A)$.