Covering Points with Affine Hyperplanes

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Joint work with Abdul Basit, Paul Horn, and Hao Huang

The Story of Jake

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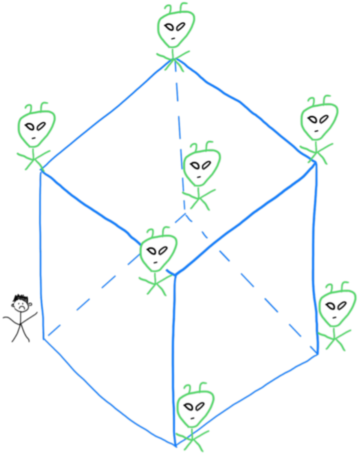
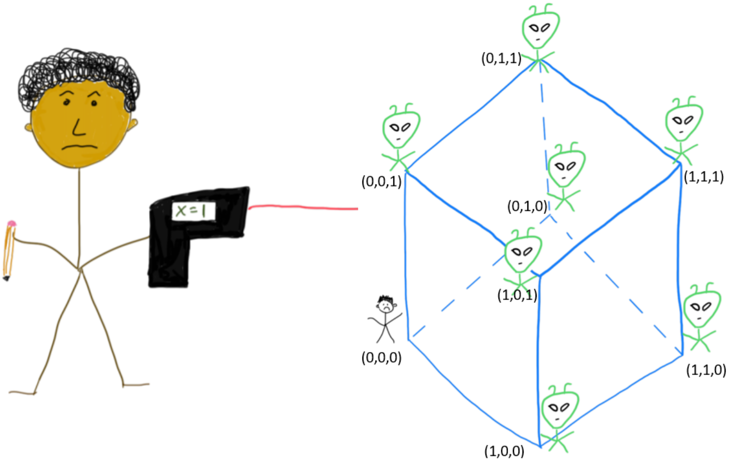


The kidnapping of Jake's advisor (artist's rendition)

The Story of Jake



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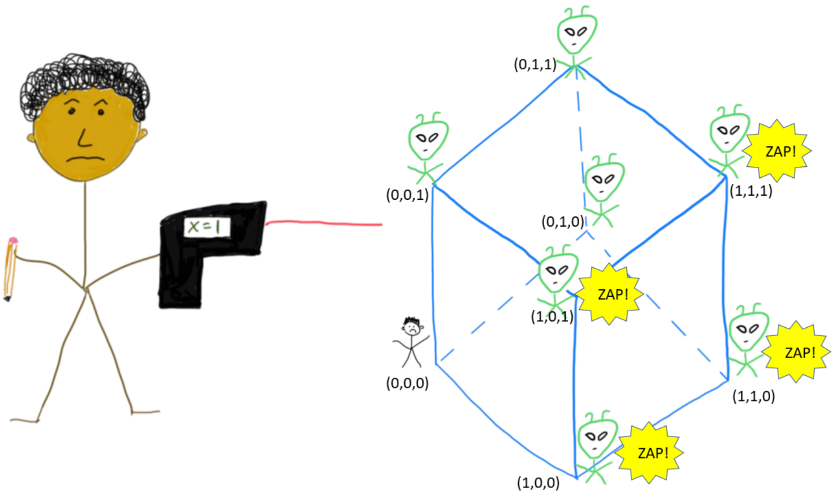

Jake's Task

Jake needs to construct a list of planes such that:

- No plane contains the origin.
- Every other point of $\{0, 1\}^3$ is contained in at least six planes.

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Jake needs to construct a list of planes such that:

- No plane contains the origin.
- Every other point of $\{0,1\}^3$ is contained in at least six planes.
- Jake can only use eleven planes!

Did he succeed?

$x = 1, y = 1, z = 1$ twice each

$x + y = 1, x + z = 1, y + z = 1$ once each

$x + y + z = 1$ twice

Question (Komjáth, 1994)

How many hyperplanes does it take to cover all the vertices of $Q^n := \{0,1\}^n$, except for one which isn't covered at all?

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How many hyperplanes does it take to cover all the vertices of $Q^n := \{0,1\}^n$, *except for one which isn't covered at all?*

A hyperplane in n-dimensional space is a flat surface with dimension $n-1$.

Example

- Lines for $n = 2$.
- Planes for $n = 3$.
- $x_1 + 2x_2 + 3x_3 - x_4 = 5$ or $3x_1 + 7x_4 = 0$ when $n = 4$.
- $a_1x_1 + a_2x_2 + \cdots + a_nx_n = c$ in general.

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How many hyperplanes does it take to cover all the vertices of $Q^n := \{0,1\}^n$*, except for one which isn't covered at all?*

Theorem (Alon–Füredi, 1993)

The minimum is n*.*

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Two of the possible constructions:

- $x_i = 1$ for $i = 1, \cdots, n$.
- $\sum_{i=1}^{n} x_i = t$ for $t = 1, \cdots, n$.

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Lower bound comes from Combinatorial Nullstellensatz.

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xy has infinitely many zeros!

Theorem (Alon, 1999)

Let F be a field and $f \in F[x_1, x_2, \cdots, x_n]$. Suppose deg $f = \sum_{i=1}^{n} t_i$ where each $t_i \geq 0$ and that $\prod_{i=1}^{n} x_i^{t_i}$ has a nonzero coefficient in f.

Then, if $S_1, S_2, \cdots, S_n$ are subsets of F with $|S_i| > t_i$, there exist $s_i \in S_i$ for $i = 1, \cdots, n$ such that $f(s_1, \cdots, s_n) \neq 0$.

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Example

Let $f(x_1, x_2) = 4x_1^2 x_2 + x_2^3 + 3x_1 x_2 - x_1 + 3$. f cannot vanish on the entirety of the grid $\{a, b, c\} \times \{d, e\}$. That is, we do not ever have

$$f(a, d) = f(a, e) = f(b, d) = f(b, e) = f(c, d) = f(c, e) = 0.$$

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Proof of Alon–Füredi result

Theorem

$n-1$ affine hyperplanes are insufficient to cover all but one vertex of Q^n.

Proof.

Assume that we can cover all but one vertex of Q^n using the hyperplanes $H_1, H_2, \cdots, H_{n-1}$. We can write H_i as $a_{i1}x_1 + \cdots + a_{in}x_n = 1$.

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Let $f = \prod_{i=1}^{n-1} P_i$ with $P_i := a_{i1}x_1 + \cdots + a_{in}x_n - 1$. f vanishes on $Q^n \setminus \{\vec{0}\}$.

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g vanishes on Q^n, contradicting Combinatorial Nullstellensatz. □

An *almost k-cover* of Q^n is a collection of affine hyperplanes which covers every point of $Q^n \setminus \{\vec{0}\}$ at least k times, without covering $\vec{0}$.

Let $f(n,k)$ be the minimum size of an almost k-cover of Q^n.

Preliminary Lower Bound

If you remove a hyperplane from an almost $k-$cover, you get an almost $(k-1)-$cover. Thus, $f(n,k) \geq f(n,k-1)+1$.

By induction, $f(n,k) \geq n+k-1$.

Preliminary Upper Bound

Use each $x_i = 1$ once for $i = 1, \cdots, n$ and use $k - t$ copies of $\sum_{i=1}^{n} x_i = t$ for $t = 1, \cdots, k-1$, for a total of $n + \binom{k}{2}$.

Example

For $n = 3, k = 4$,

- $x_1 = 1, x_2 = 1, x_3 = 1$
- $x_1 + x_2 + x_3 = 1$ three times
- $x_1 + x_2 + x_3 = 2$ twice
- $x_1 + x_2 + x_3 = 3$

Improvements

Let $f(n, k)$ be the minimum size of an almost k-cover of Q^n.

Theorem (C.–Huang, 2020)

For $n \geq 2$,

$$f(n, 3) = n + 3.$$

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Theorem (Sauermann–Wigderson, 2022)

For $n \geq 2k-3,\ k \geq 2$,

$$f(n,k) \geq n+2k-3.$$

Upper Bound

Conjecture (C.–Huang, 2020)

For each k*,* $f(n,k) = n + \binom{k}{2}$ *for sufficiently large* n*.*

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For each k, $f(n, k) = n + \binom{k}{2}$ for sufficiently large n.

Theorem (Alon, 2020; C., 2024++)

Any almost k-cover containing at least $n - 2$ of the hyperplanes $x_1 = 1, x_2 = 1, \ldots, x_n = 1$ must have size at least $n + \binom{k}{2}$.

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Example (C.–Grzesik–Kim, 2023++)

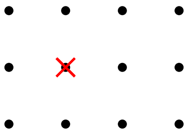
- $x_1 = 1$
- $x_1 + x_j = 1$ for $j = 2, \dots, n$
- $k - m$ copies of $(2 - n/m)x_1 + (x_2 + \cdots + x_n)/m = 1$ for $m = 1, \dots, k-1$

Almost Covers of Rectangular Grids

Theorem (Alon–Füredi, 1993)

For sets $S_1, S_2, \cdots, S_n \subset \mathbb{R}$, the minimum number of affine hyperplanes in $\mathbb{R}^n$ needed to cover all but one point of $S_1 \times S_2 \times \cdots \times S_n$ and leave the last point uncovered is

$$\sum_{i=1}^{n}(|S_i| - 1).$$



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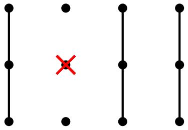

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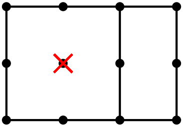
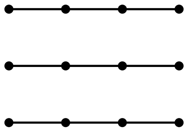


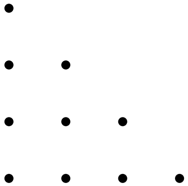
Why remove a point?

If we instead insist on covering every point of $S_1 \times S_2 \times \dots S_n$, then this is a very boring question.



Every point lies on a hyperplane of maximum size!

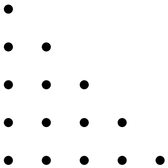
Triangular Grids



Not every point lies on a hyperplane of maximum size!

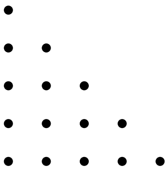
Notation

Let $T_d(n) := \{(x_1, \cdots, x_d) \in \mathbb{Z}^d_{\geq 0} \mid x_1 + \cdots + x_d \leq n - 1\}$.



Let $f(n, d, k)$ denote the minimum number of hyperplanes needed to cover every point of $T_d(n)$ at least k times.

Integer Covering



Theorem (Basit–C.–Horn, 2023+)

For all $n \geq 2$,

$$f(n,2,k) = \begin{cases} n & \text{if } k = 1, \\ \lceil 3n/2 \rceil & \text{if } k = 2, \\ \lceil 9n/4 \rceil & \text{if } k = 3, \\ 3n & \text{if } k = 4. \end{cases}$$

Proof for $k = 4$: Upper Bound

Theorem (Basit–C.–Horn, 2023+)

For all $n \geq 2$, $f(n, 2, 4) = 3n$.

Proof.

Our construction only uses lines parallel to the sides of the outer triangle.

- Lines $x = i, y = i$, and $x + y = n - 1 - i$ for $i \in \{0, \ldots, \lfloor \frac{n-1}{3} \rfloor\}$ have multiplicity 2.
- Lines $x = i, y = i$, and $x + y = n - 1 - i$ for $i \in \{\lfloor \frac{n-1}{3} \rfloor + 1, \ldots, \lfloor \frac{2n}{3} \rfloor - 1\}$ have multiplicity 1.

$\square$

Proof for $k = 4$: Lower Bound

Theorem (Basit–C.–Horn, 2023+)

For all $n \geq 2$, $f(n, 2, 4) = 3n$.

Proof.

We show $f(n, 2, 4) \geq 3n$ by induction.

If we use an outer line ($x = 0$, $y = 0$, or $x + y = n - 1$) at least three times, then we require at least $f(n - 1, 2, 4) + 3 = (3n - 3) + 3$ lines.

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Using each outer line twice leaves $3(n - 2)$ points on the boundary that need to be covered an additional two times each. Only two of these can be covered at a time by any other line, forcing at least $\frac{3(n-2)(2)}{2} = 3n - 6$ more lines. □

Integer Program

$f(d, n, k)$ is the minimum number of hyperplanes needed to cover every point of $T_d(n)$ at least k times each.

This is the optimum of an integer program:

- Variables correspond to how many times each hyperplane is used.
- Constraints correspond to each grid point being covered at least k times.

Linear Relaxation

We define $f^*(n, d, k)$ to be the optimum of the linear relaxation. We write $f^*(n, d) := f^*(n, d, 1)$.

$f(n, d, k) \geq f^*(n, d, k) = kf^*(n, d)$.

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Theorem (Basit–C.–Horn, 2023+)

For all integers $j \geq 0$,

$$\begin{cases} f^*(3j+1,2) = 2j+1, \\ f^*(3j+2,2) = 2j+1+\dfrac{2j+1}{3j+2}, \\ f^*(3j+3,2) = 2j+2+\dfrac{j+1}{3j+4}. \end{cases}$$

$$1, \frac{3}{2}, \frac{9}{4}, 3, \frac{18}{5}, \frac{30}{7}, 5, \ldots$$

Fractional Covering: Upper Bound

Theorem (Basit–C.–Horn, 2023+)

$f^*(3j+1, 2) = 2j+1$ *for all integers* $j \geq 0$.

$T_2(3j+1) = \{(x, y) \mid x, y \geq 0, x + y \leq 3j\}$. We can cover all these points with the following lines:

- $x = i$ for $i = 0, \cdots, 2j-1$ with weight $\frac{2j-i}{3j}$,
- $y = i$ from $i = 0, \cdots, 2j-1$ with weight $\frac{2j-i}{3j}$, and
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If $i_1, i_2 \leq 2j-1$, (i_1, i_2) is covered with weight $\frac{2j-i_1}{3j}$ by a vertical line and weight $\frac{2j-i_2}{3j}$ by a horizontal line for a total weight of $\frac{4j-i_1-i_2}{3j}$.

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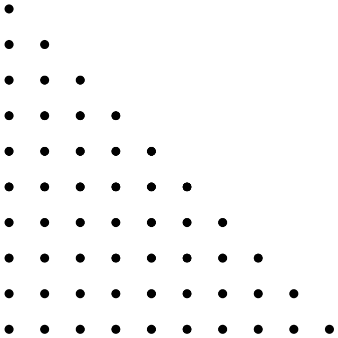
If this is not at least 1, $i_1 + i_2 \geq j+1$ and the point is covered by a diagonal line with weight $\frac{i_1+i_2-j}{3j}$ for a total weight of 1.

Fractional Covering: Lower Bound

Theorem (Basit–C.–Horn, 2023+)

$f^*(3j+1,2) = 2j+1$ *for all integers* $j \geq 0$.



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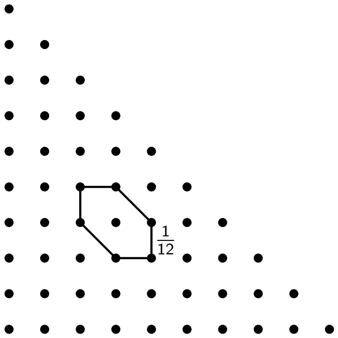

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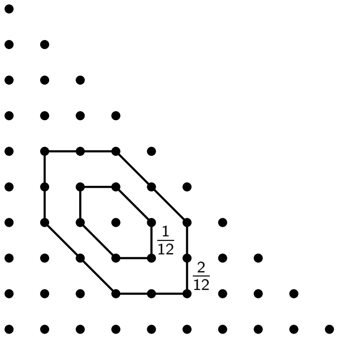

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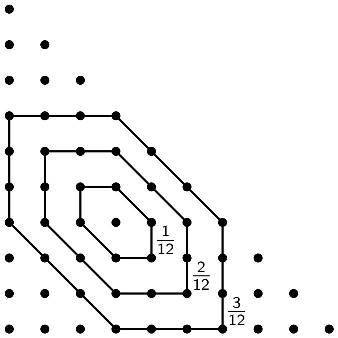

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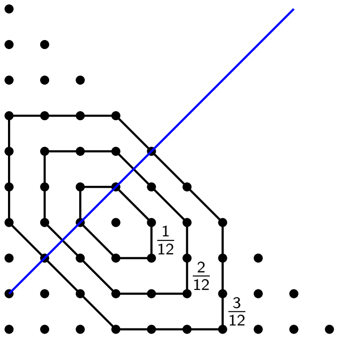

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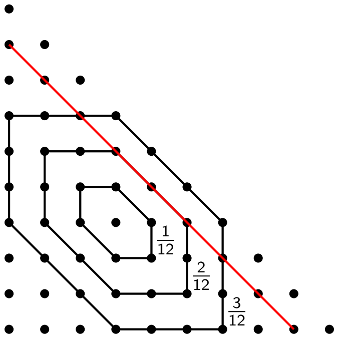

We automatically get the bound $f(n, 2, k) \geq kf^*(n, 2)$ but it is not tight.

For example, $f^*(n, 2) = 2n/3 + O(1)$, but $f(n, 2, 4) = 3n$ rather than $8n/3 + O(1)$.

Computations suggest $f(n, 2, k) = C_k n + O(1)$ for some constant C_k and in particular that $C_5 = 18/5$, $C_6 = 30/7$, and $C_7 = 5$.

Conjecture

Conjecture (Basit–C.–Horn, 2023+)

For $k \geq 1$,

$$f(n,2,k) = (f^*(k,2))n + O_k(1).$$

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For $k \geq 1$,

$$f(n, 2, k) = (f^*(k, 2))n + O_k(1).$$

- We can translate the upper bound construction for the fractional problem to the necessary upper bound construction for the integer program.
- The desired lower bound on $f(n, 2, k)$ holds under certain natural constraints.

Open Problems

- Determine the asymptotic formula (in terms of n) for general $f(n, d, k)$.
- Is $f(n, d, k) \geq f^*(k, d)n$ for all n, d, k?

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- Does $f(n, d, k) = f(k, d, n)$ for all n, d, k?

Díky moc!

Recall that $f(n, d, k)$ is the minimum number of hyperplanes needed to cover every point of $T_d(n) := \{(x_1, \cdots, x_d) \in \mathbb{Z}^d_{\geq 0} \mid x_1 + \cdots + x_d \leq n - 1\}$ at least k times.

Theorem

a) *If $k \geq 2$ and $d \geq 2k - 3$, then*

$$f(n, d, k) = \left(1 + \frac{k-1}{d-k+2}\right) n + O_{d,k}(1),$$

b) *If $k \geq 3$ and $2k - 3 \geq d \geq k - 2$, then*

$$f(n, d, k) = \left(2 + \frac{2k-3-d}{2d+3-k}\right) n + O_{d,k}(1).$$

Fix d and k.

1) Suppose you want to show a lower bound of $f(n, d, k) \geq Cn + C'$ via induction on n. It suffices to assume that all *bounding hyperplanes* ($x_i = 0$ or $x_1 + \cdots + x_d = n - 1$) are used fewer than C times.

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2) The intersection of a bounding hyperplane H with $T_d(n)$ is a copy of $T_{d-1}(n)$. Any hyperplane not parallel to H intersects this in an affine subspace of dimension $d - 2$. Thus, the number of hyperplanes needed to cover k times this copy of $T_{d-1}(n)$ without using H is at least $f(n, d - 1, k)$.

We induct on k. Suppose we wish to show that $f(n, 6, 4) = 7n/4 + O(1)$ and we already know that $f(n, 5, 3) = 3n/2 + O(1)$.

Proof Example

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By Observation 1), it suffices to assume that every bounding hyperplane of $T_6(n)$ has multiplicity at most 1. Then excluding the bounding hyperplanes used, each face of the grid, which is a copy of $T_5(n)$, includes an interior copy of $T_5(n-6)$ whose points have been covered at most once.

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We cannot use anymore bounding hyperplanes so by Observation 2), each of these copies requires at least $f(n-6,5,3) = 3n/2 + O(1)$ hyperplanes to be covered an additional three times. However, no hyperplane will intersect all seven copies of $T_5(n-6)$ that need to be covered, so this requires at least

$$\binom{7}{6}(3n/2 + O(1)) = 7n/4 + O(1).$$