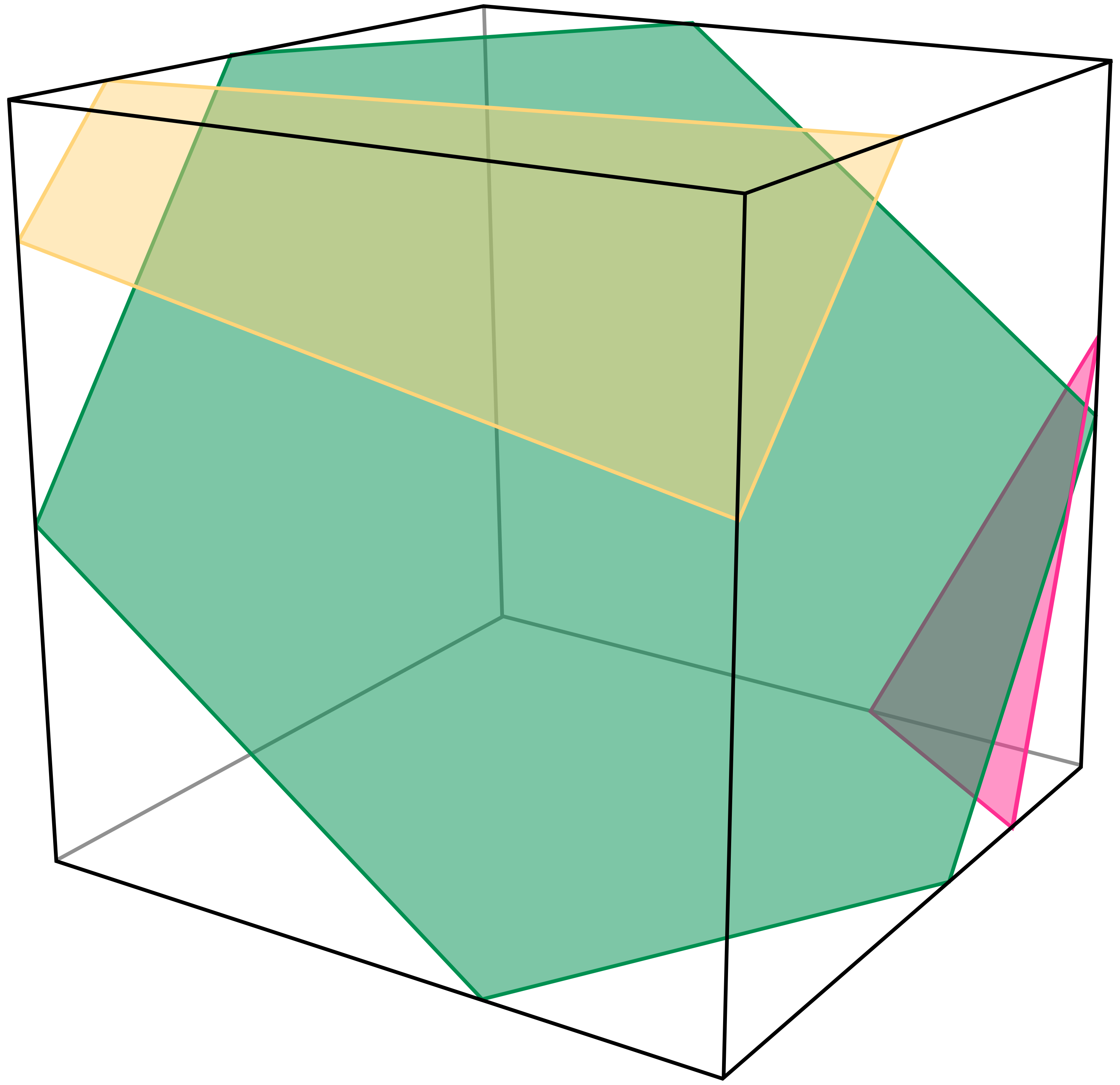
SLICES OF HYPERCUBES



Chiara Meroni

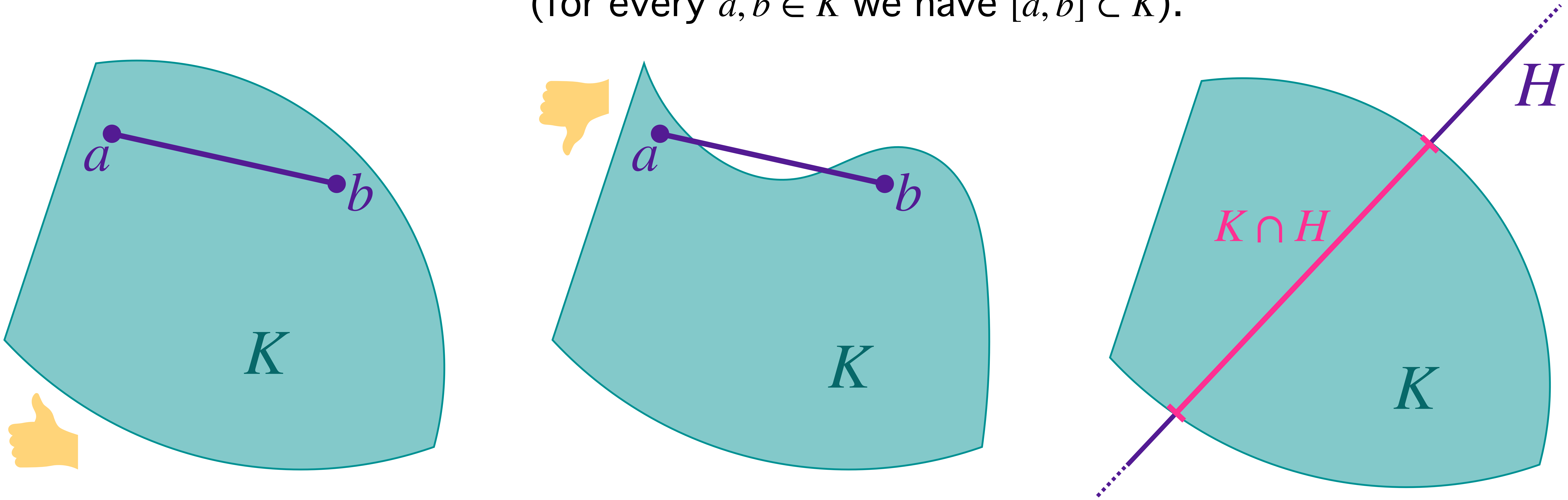
ETH zürich
Institute for Theoretical Studies

Algebra colloquium

Charles University, April 2025

Convex bodies and slices

Definition: K in $\mathbb{R}^d$ is a convex body if it is compact and convex (for every $a, b \in K$ we have $[a, b] \subset K$).



Definition: a slice of a convex body K in $\mathbb{R}^d$ the intersection $K \cap H$ with a hyperplane

$$H = \{x \in \mathbb{R}^d \mid c_0 + c_1 x_1 + \ldots + c_d x_d = 0\}.$$

Busemann–Petty problem 1956 $\longrightarrow$ 1999

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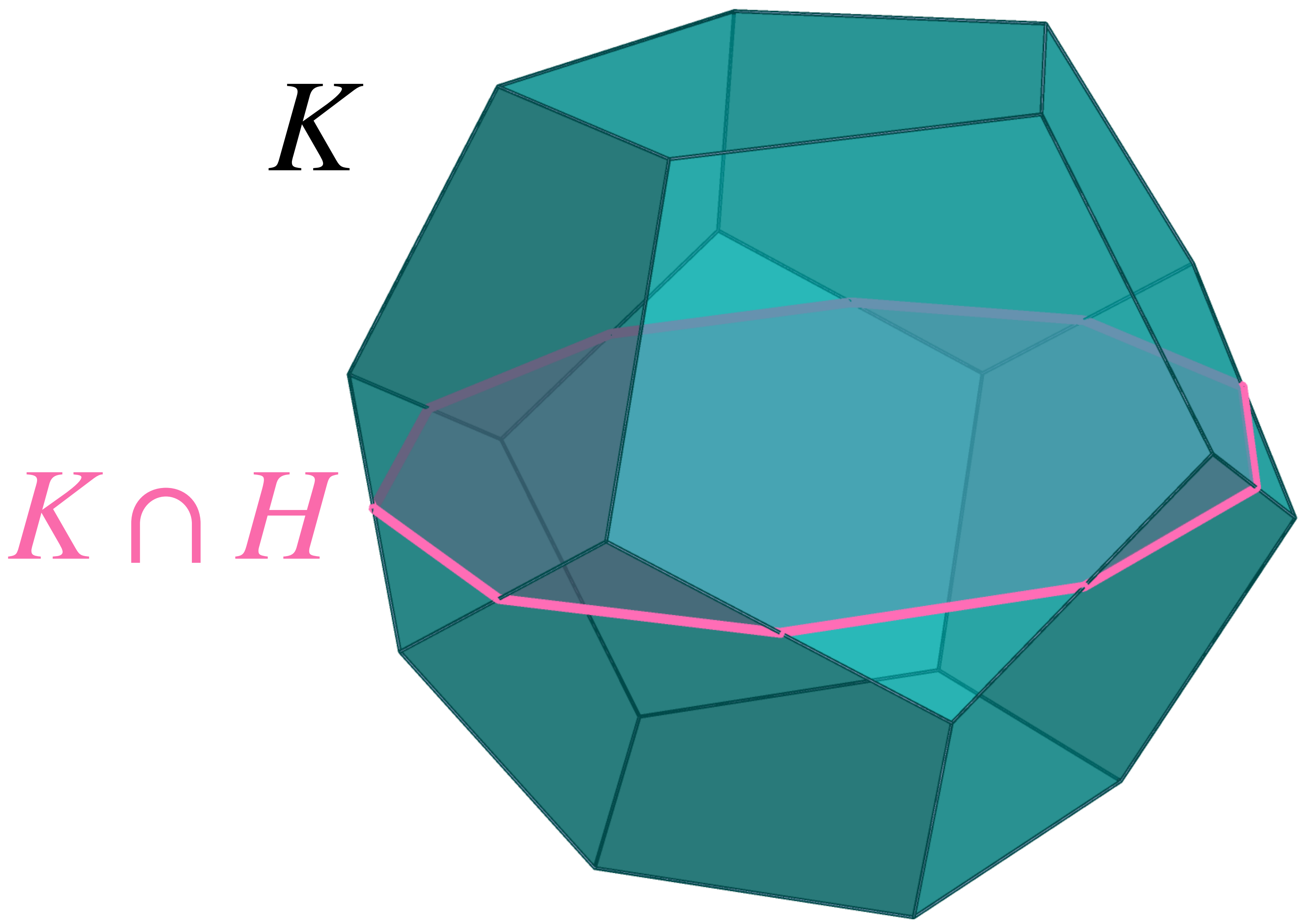
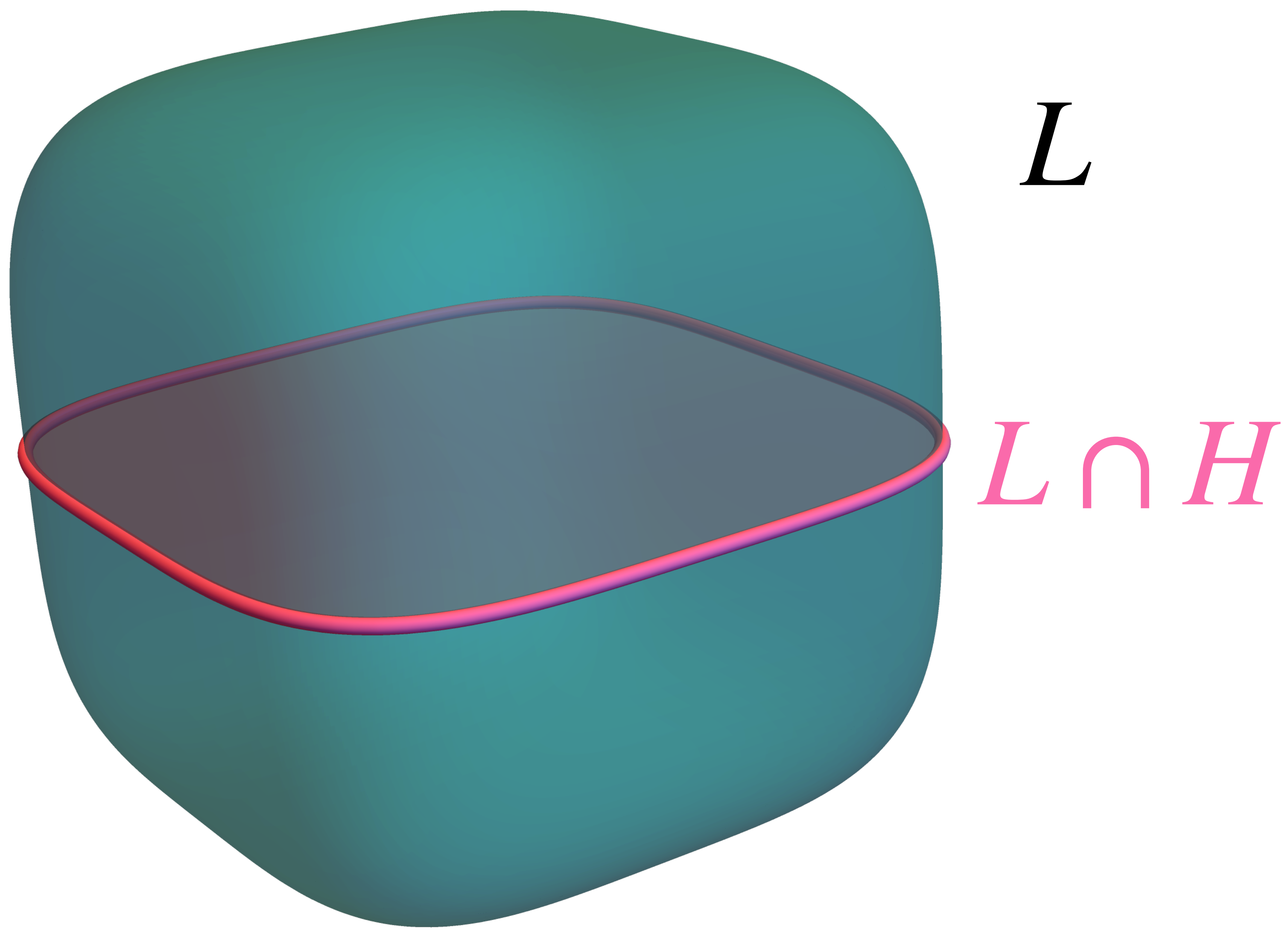
Let K and L be convex bodies in $\mathbb{R}^d$. Assume that for every hyperplane H through the origin,

$$\mathrm{vol}\,(K \cap H) \leq \mathrm{vol}\,(L \cap H).$$





Does this imply that $\mathrm{vol}\,(K) \leq \mathrm{vol}\,(L)$?

Busemann–Petty problem

$\mathrm{vol}(K) \leq \mathrm{vol}(L)$?

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In general NOT true!

It holds for convex bodies in $\mathbb{R}^d$ when $d \leq 4$.

Let's refine it:

Bourgain's slicing conjecture (1986):

$\mathrm{vol}(K) \leq C\ \mathrm{vol}(L)$,

where C does $\boxed{\text{not}}$ depend on d.

Rephrase:

Bourgain's slicing conjecture (1986):

Let $K \subset \mathbb{R}^d$ be a convex body of volume 1.
Does there exist a hyperplane H satisfying

$$\mathrm{vol}(K \cap H) > \frac{1}{C}$$

where C does $\boxed{\text{not}}$ depend on d?

arXiv > math > arXiv:2412.15044

Affirmative Resolution of Bourgain's Slicing Problem using Guan's Bound

Boaz Klartag, Joseph Lehec

Building upon

arXiv > math > arXiv:2412.09075

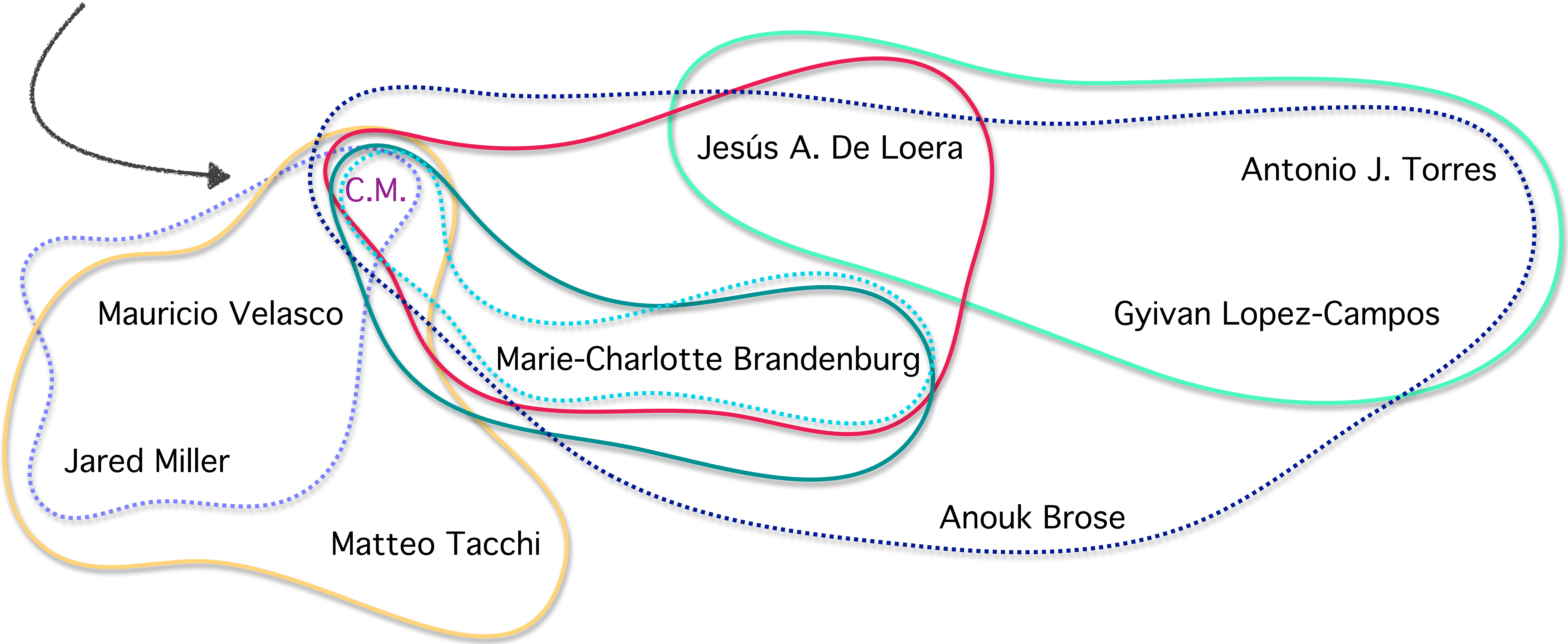
A note on Bourgain's slicing problem

Qingyang Guan

Best slices

This motivates the study of extremal slices of convex bodies

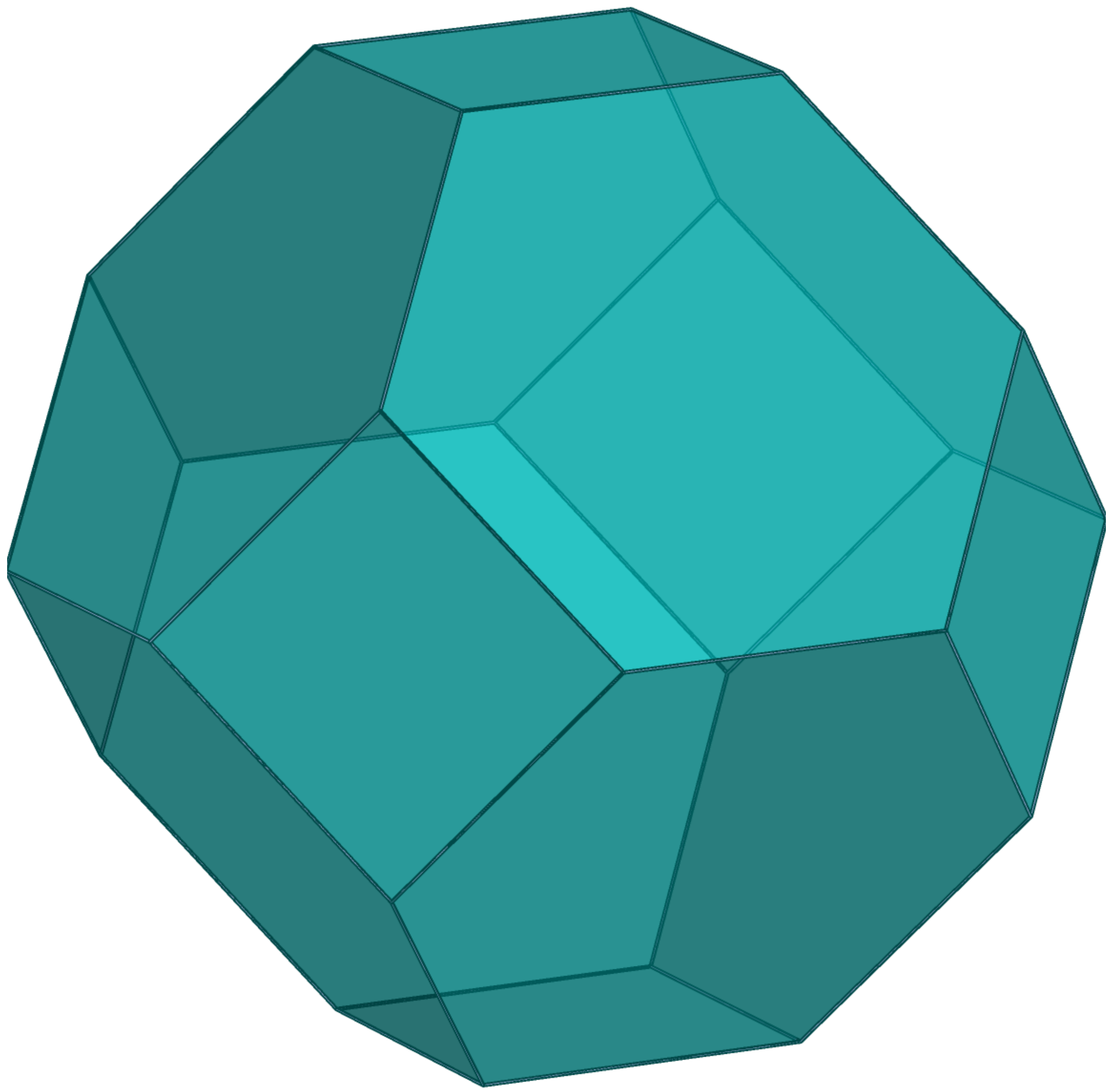
Our main interest: doing (different types of) optimization over slices



...and many more

Goal: find the "best" slice of a polytope

$P =$ permutahedron



Goal: find the "best" slice of a polytope

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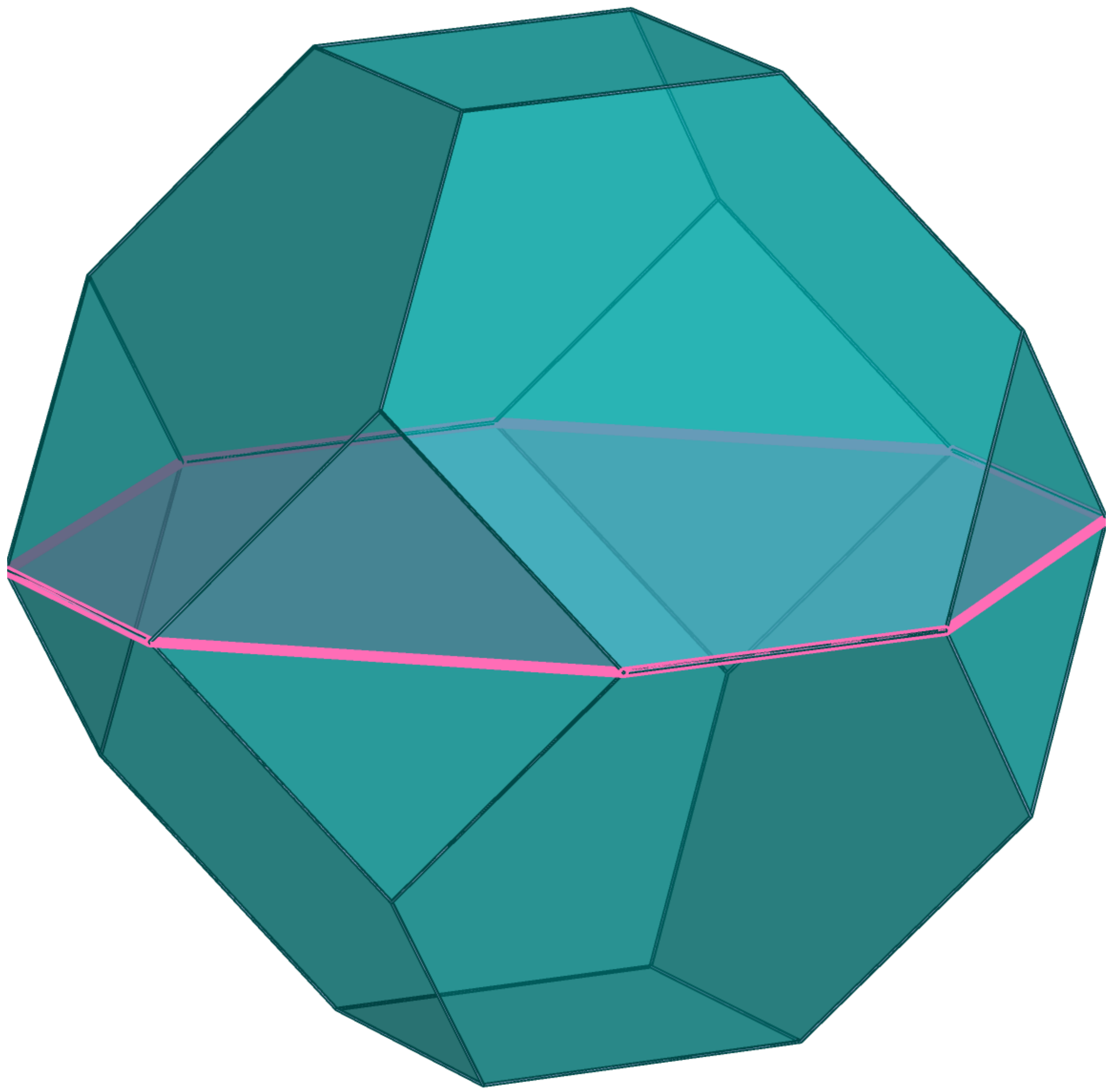
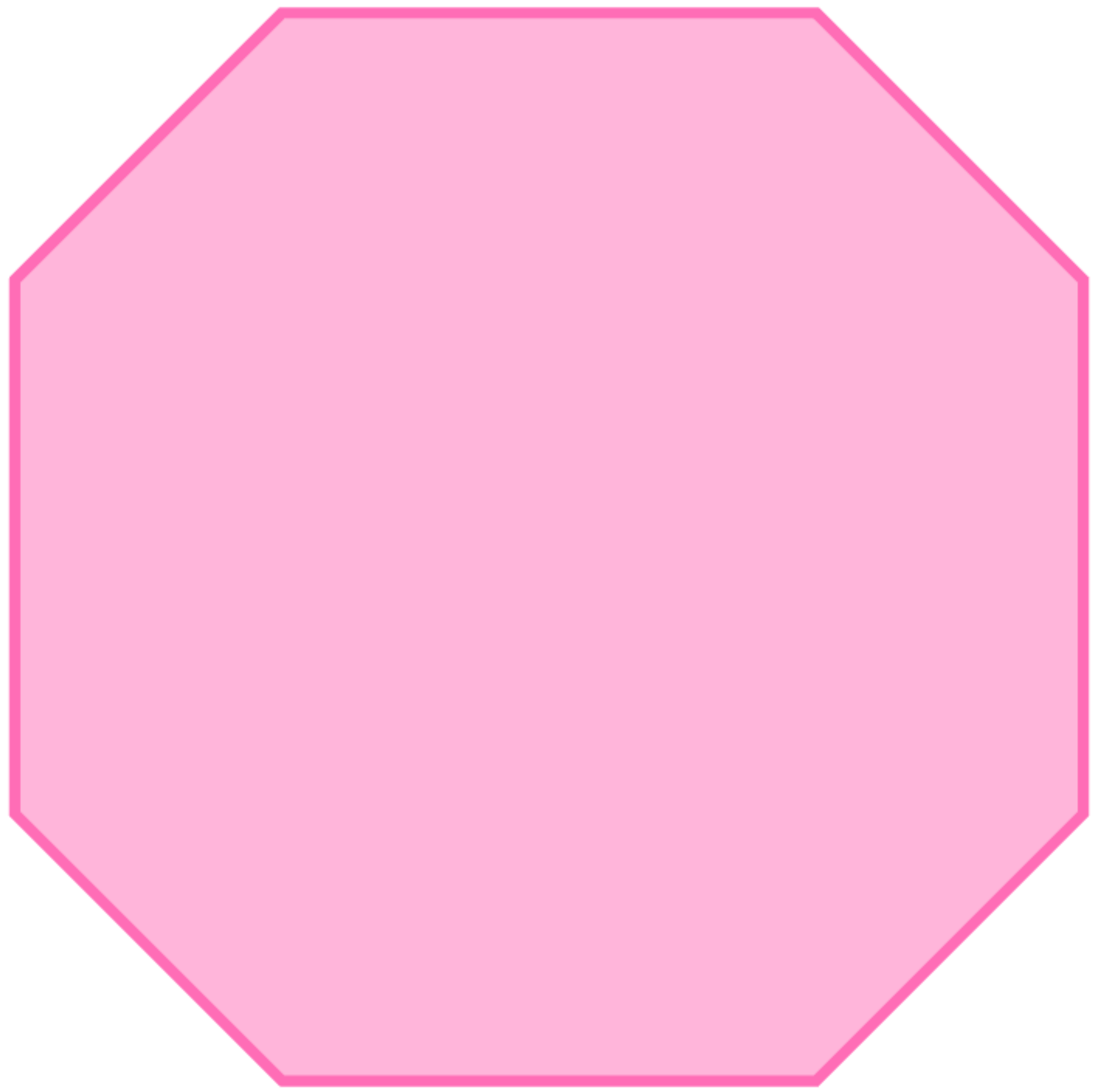
P = permutahedron

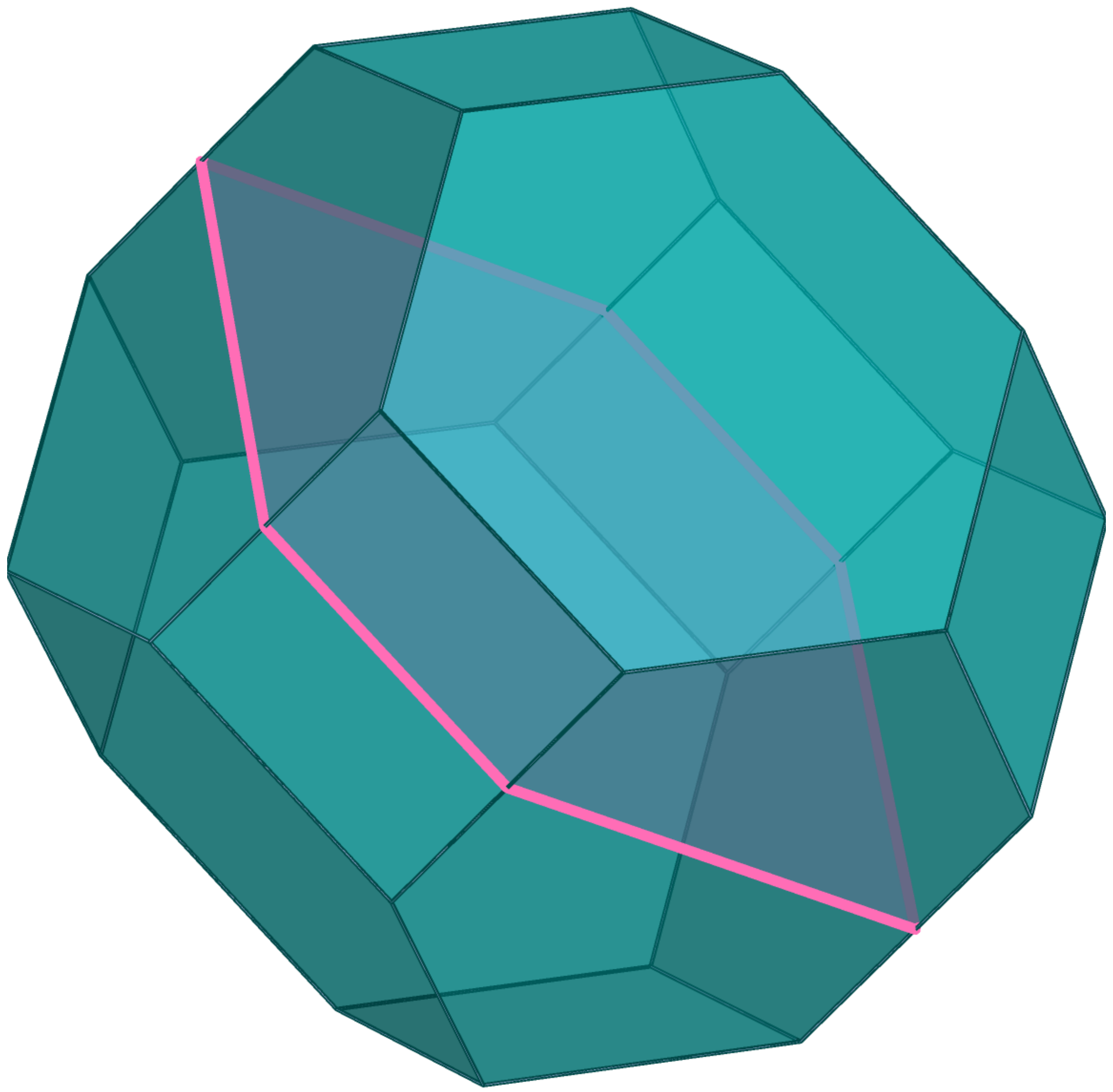
best = largest volume



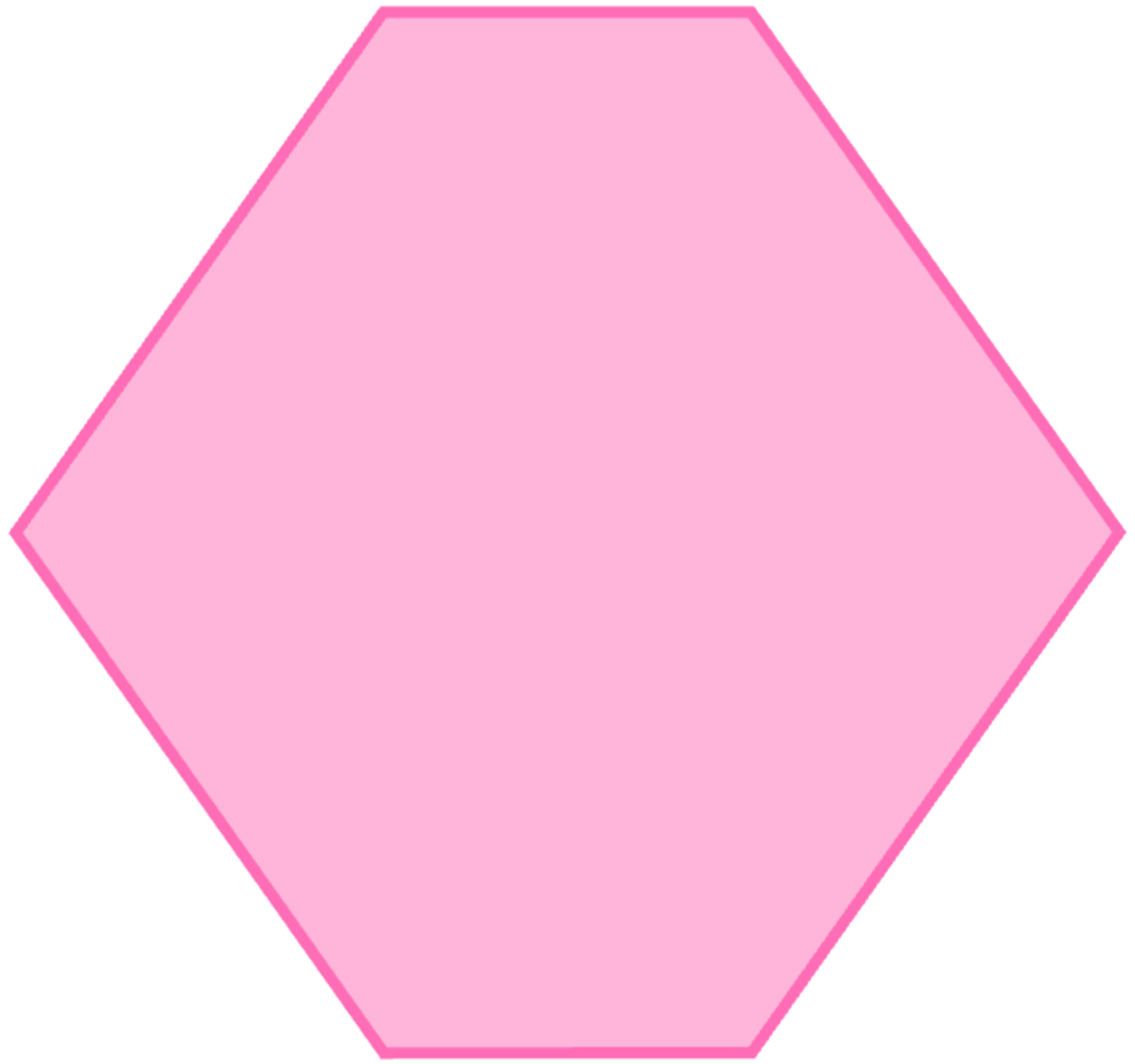


Goal: find the “best” slice of a polytope

P = permutahedron



best = smallest volume



Goal: find the "best" slice of a polytope

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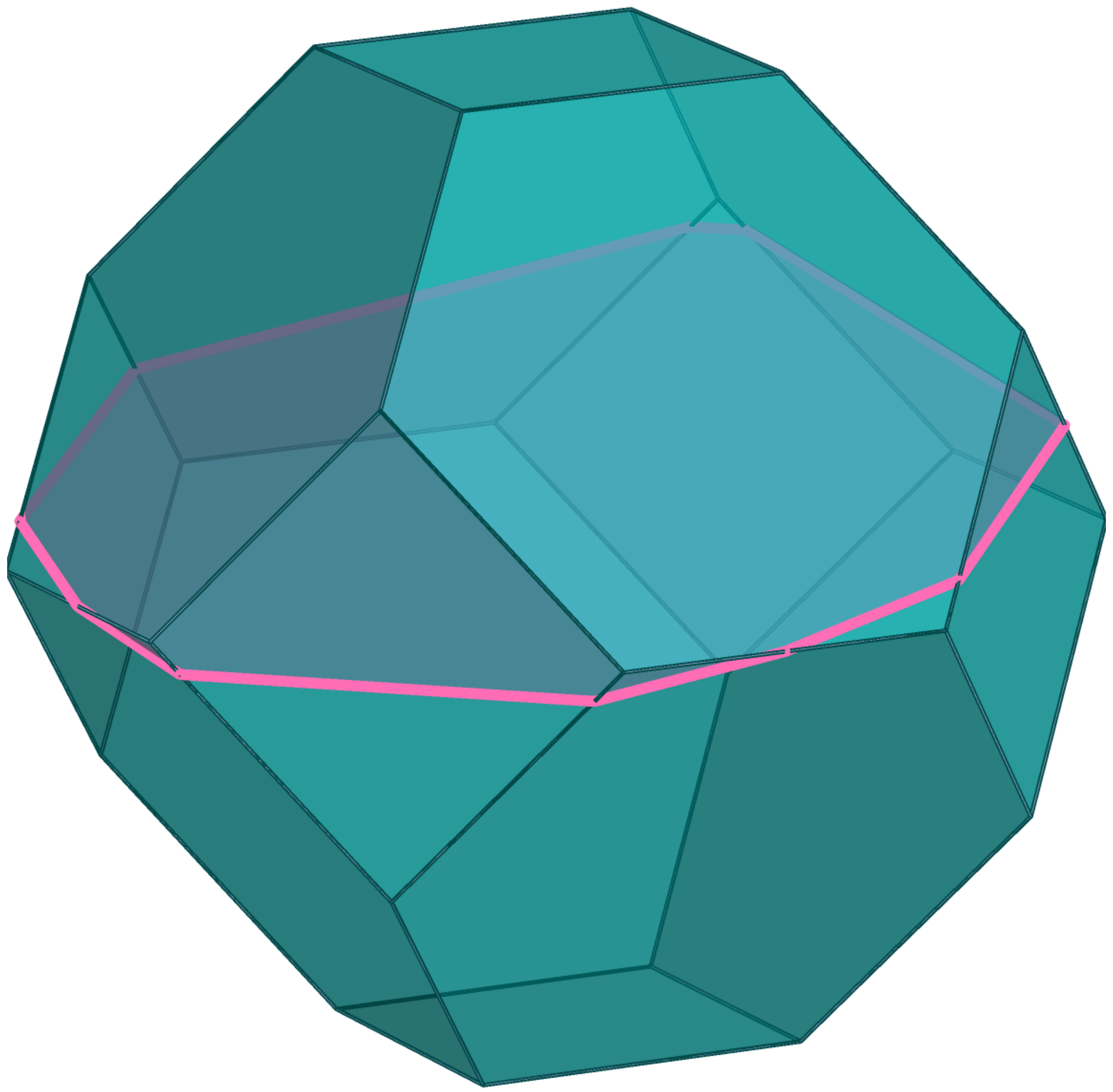
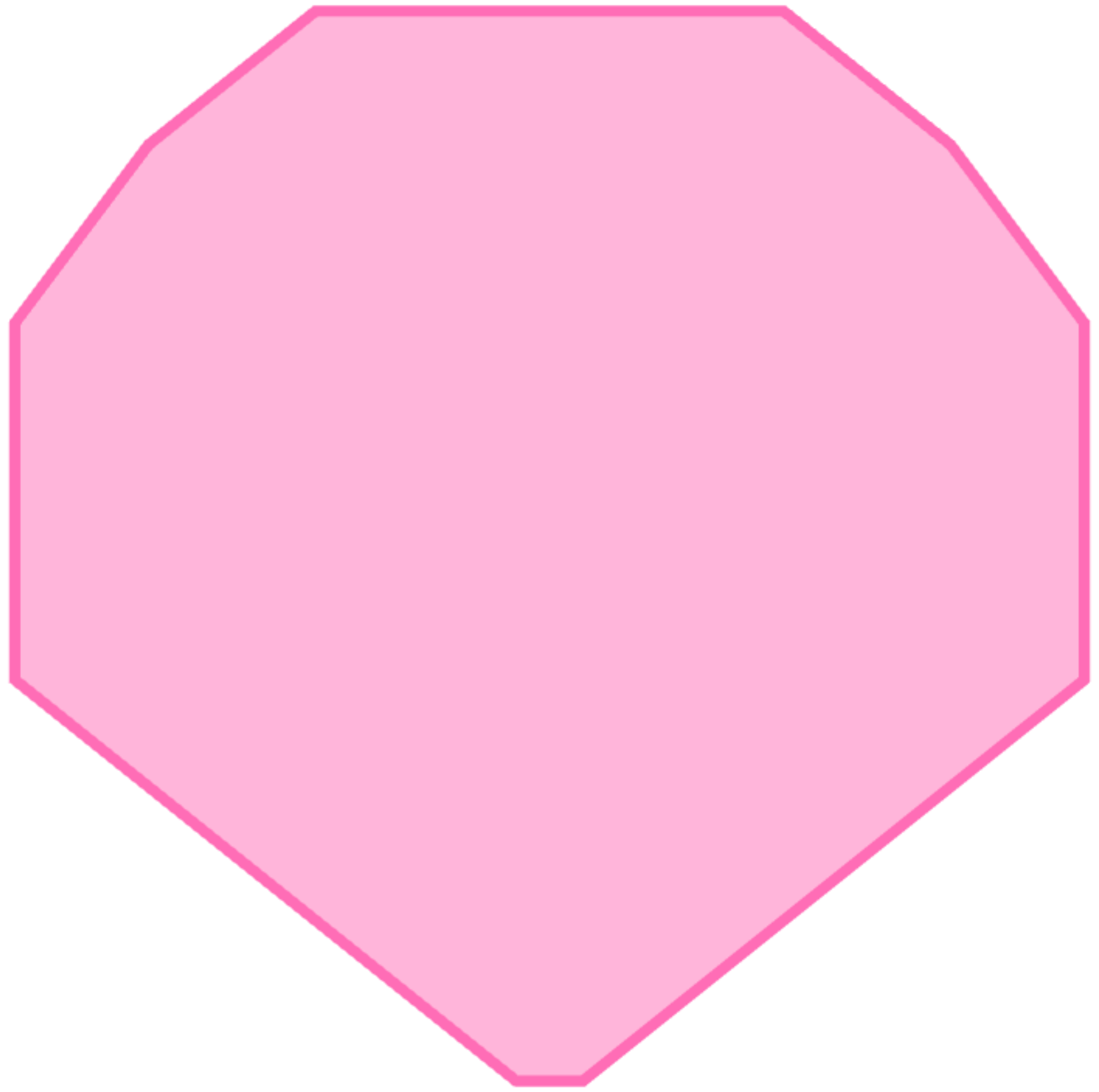
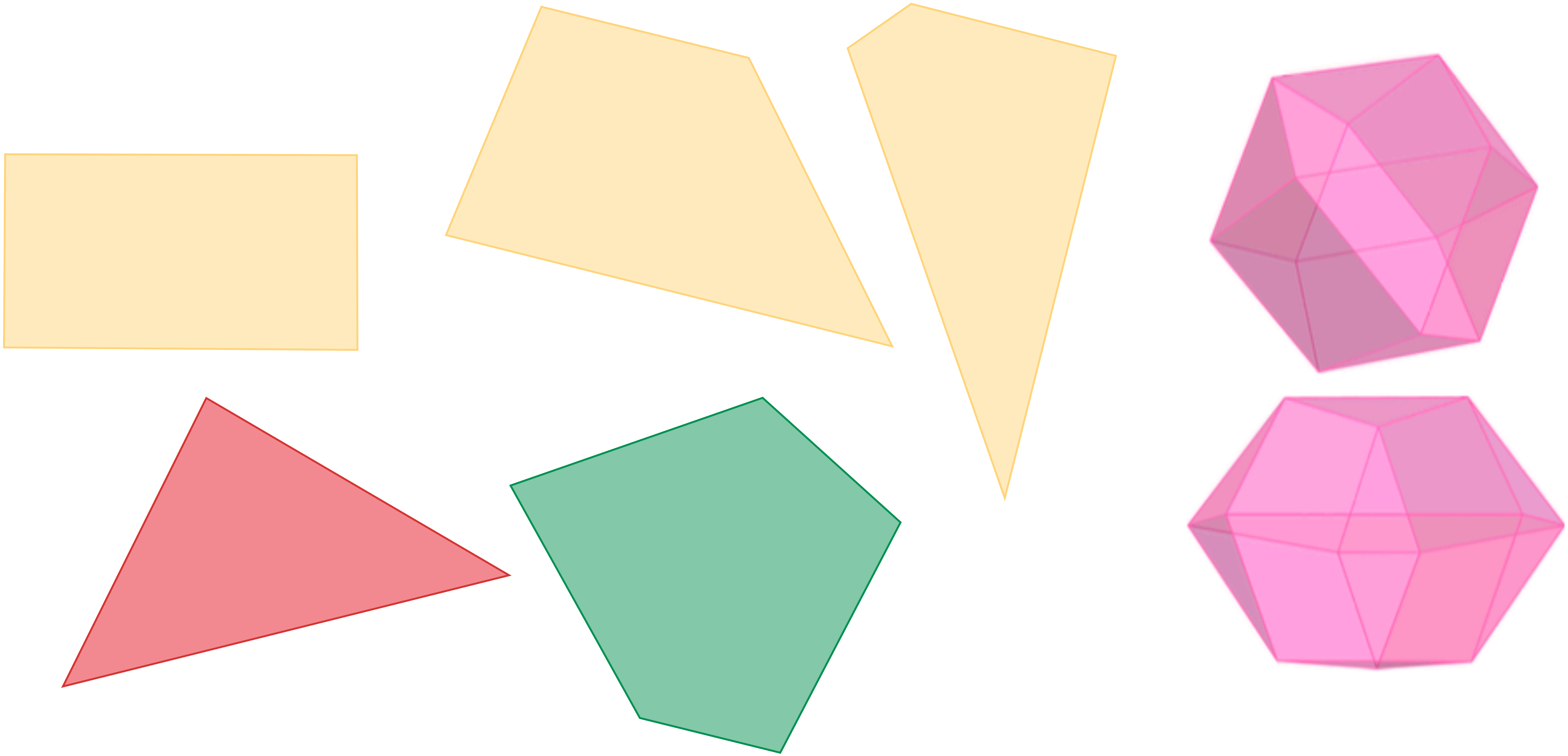
P = permutahedron



best = largest #vertices

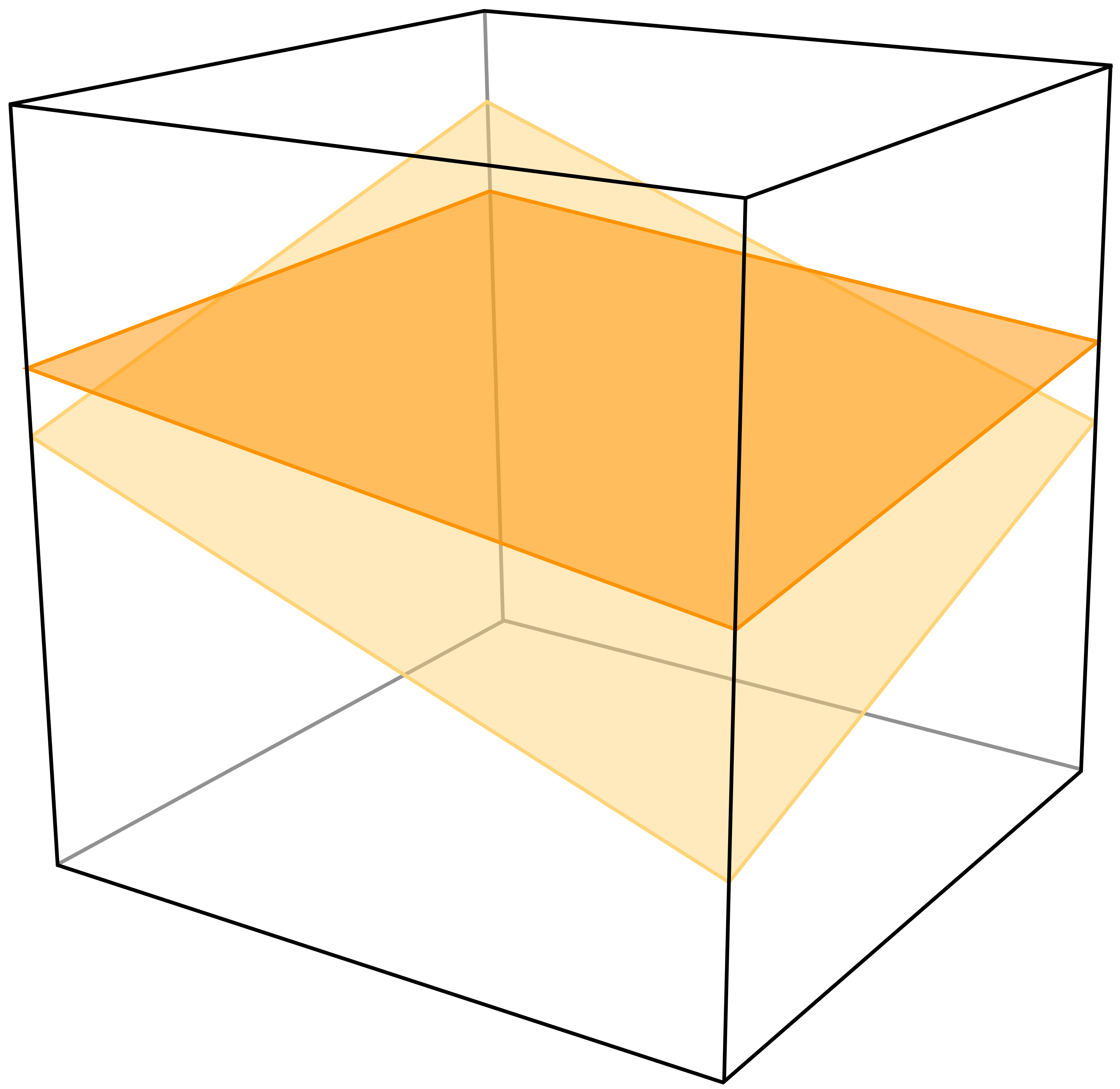


Main idea: combinatorial type

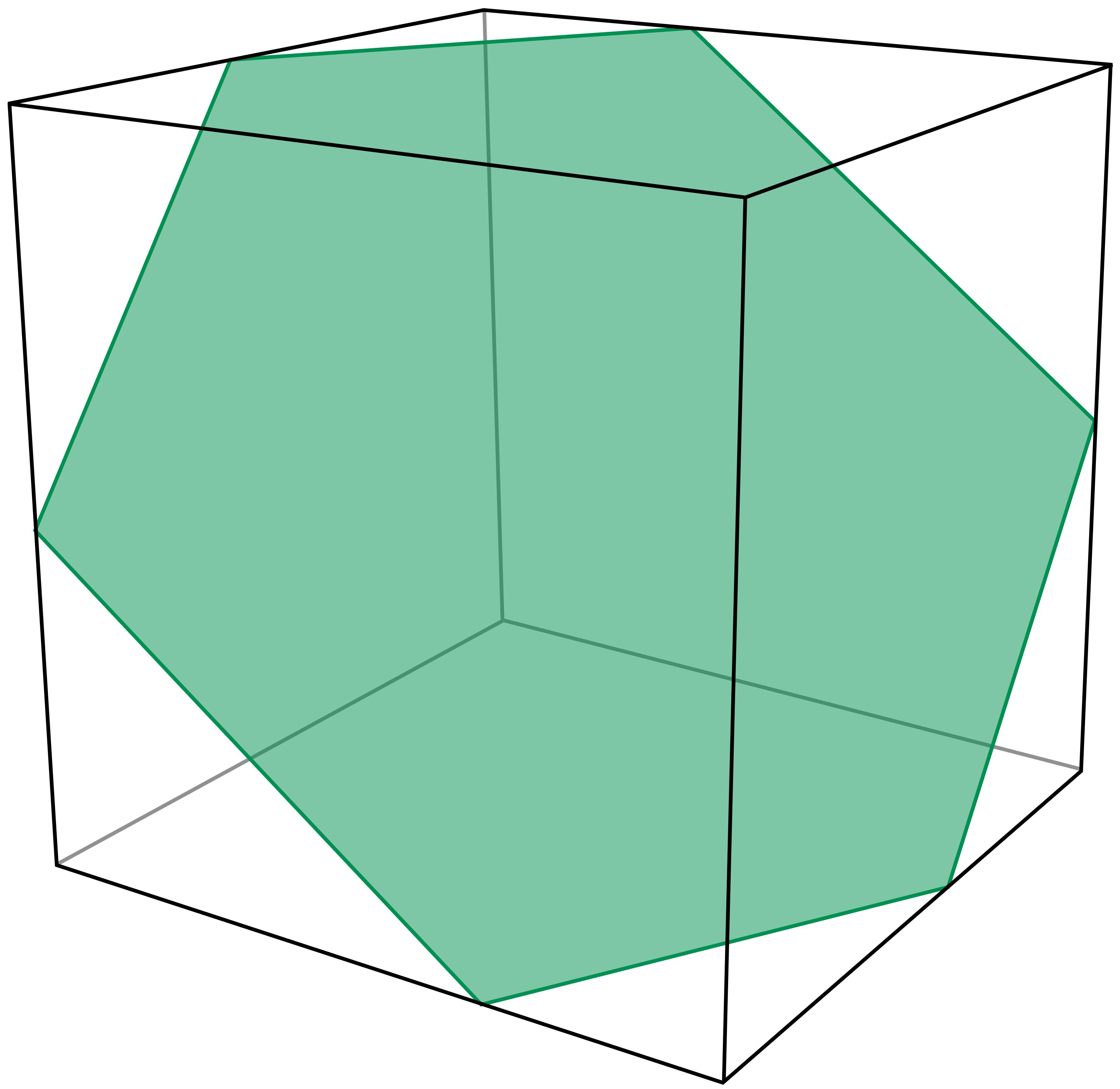


Main idea

$P =$ cube



Small perturbations (generically) preserve the combinatorial type



The combinatorial type (possibly) changes when the hyperplane crosses a vertex

Fact: if H intersects the same edges of P, then $K \cap H$ has the same combinatorial type

Not an “ if and only if ”

Fact: $K \cap H_1$ and $K \cap H_2$ can have the same combinatorial type even if H_1 and H_2 intersect different edges.

Hyperplane arrangements

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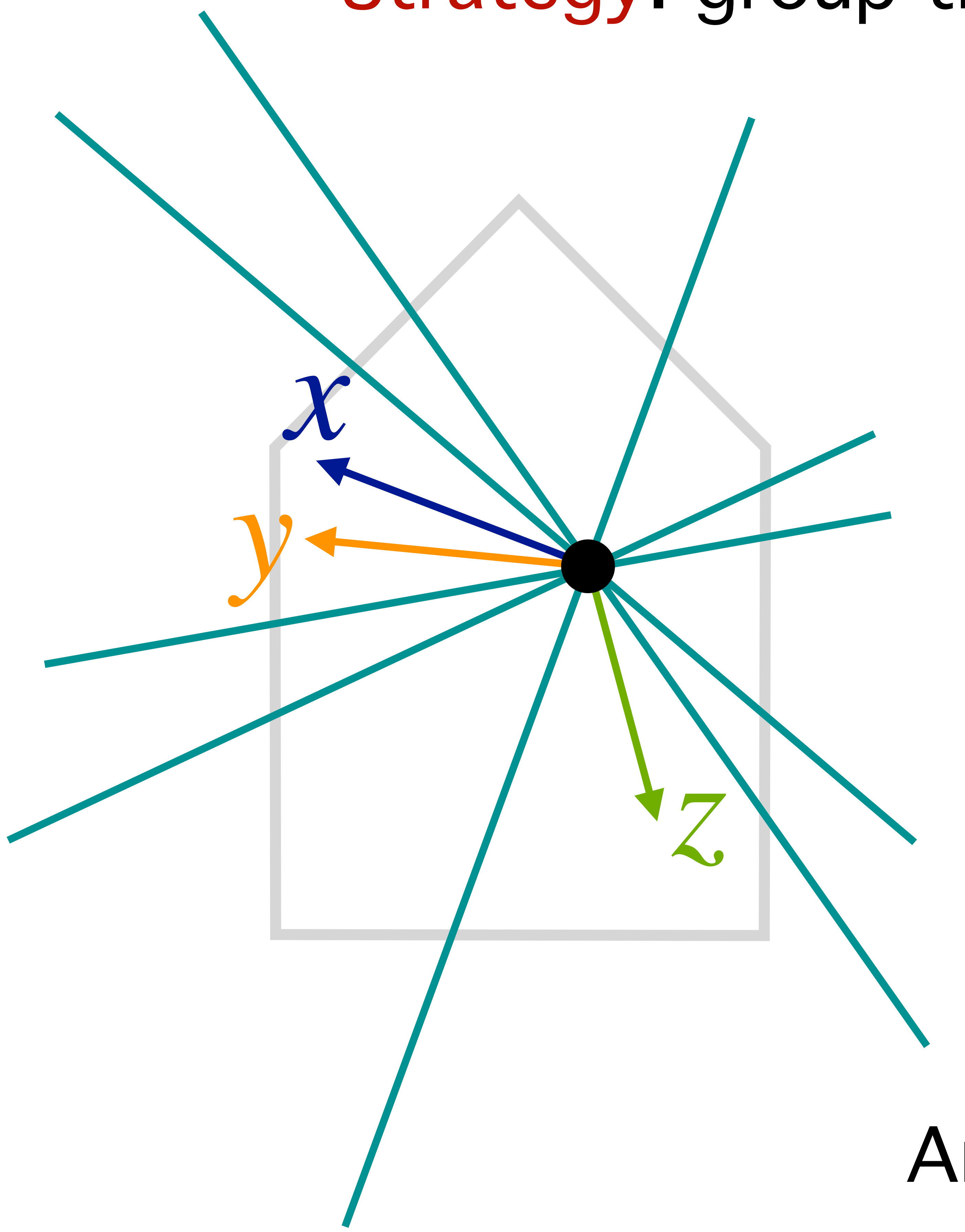
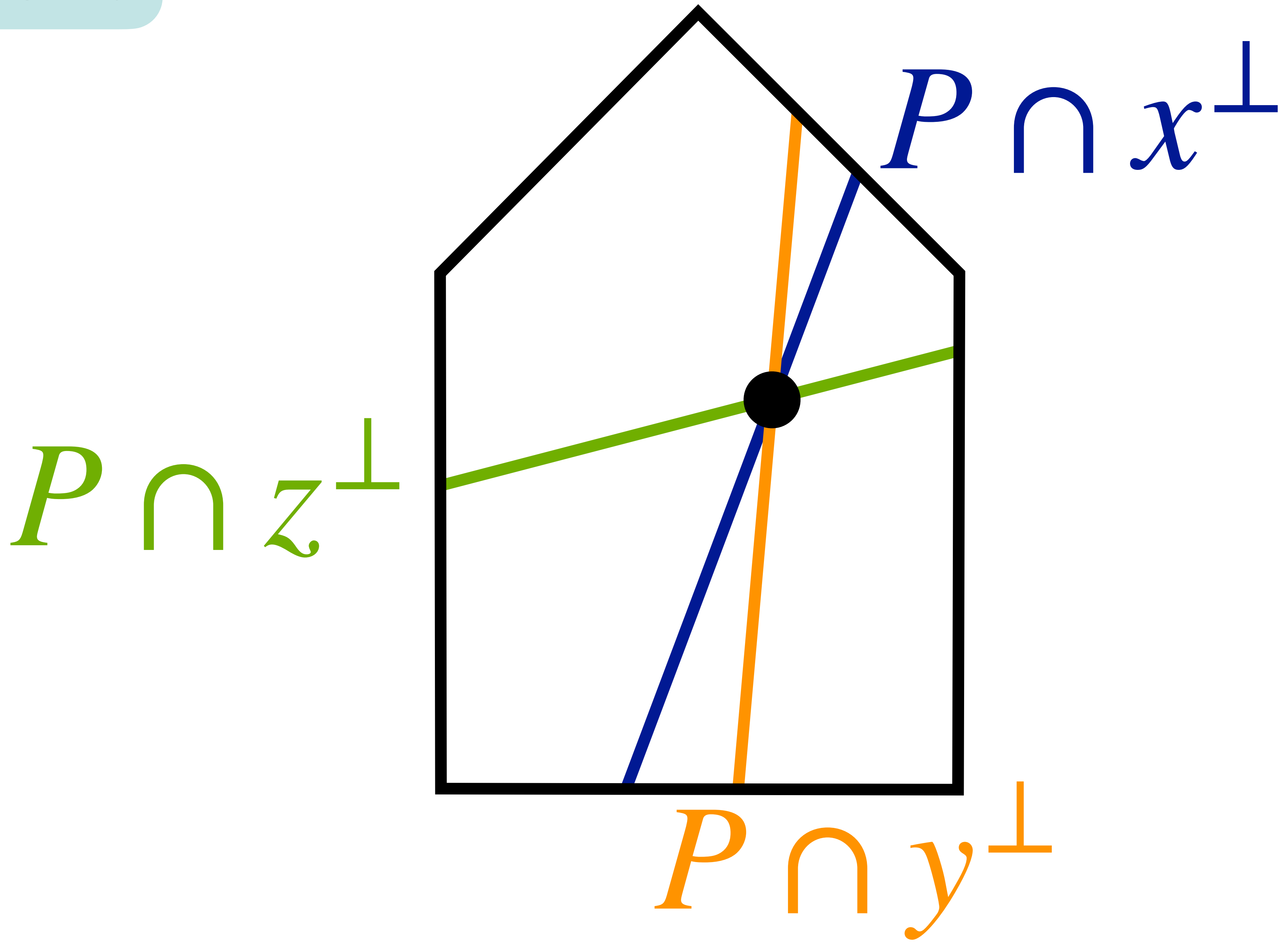
Strategy: group the slices that intersect the same sets of edges of P

Hyperplane arrangement

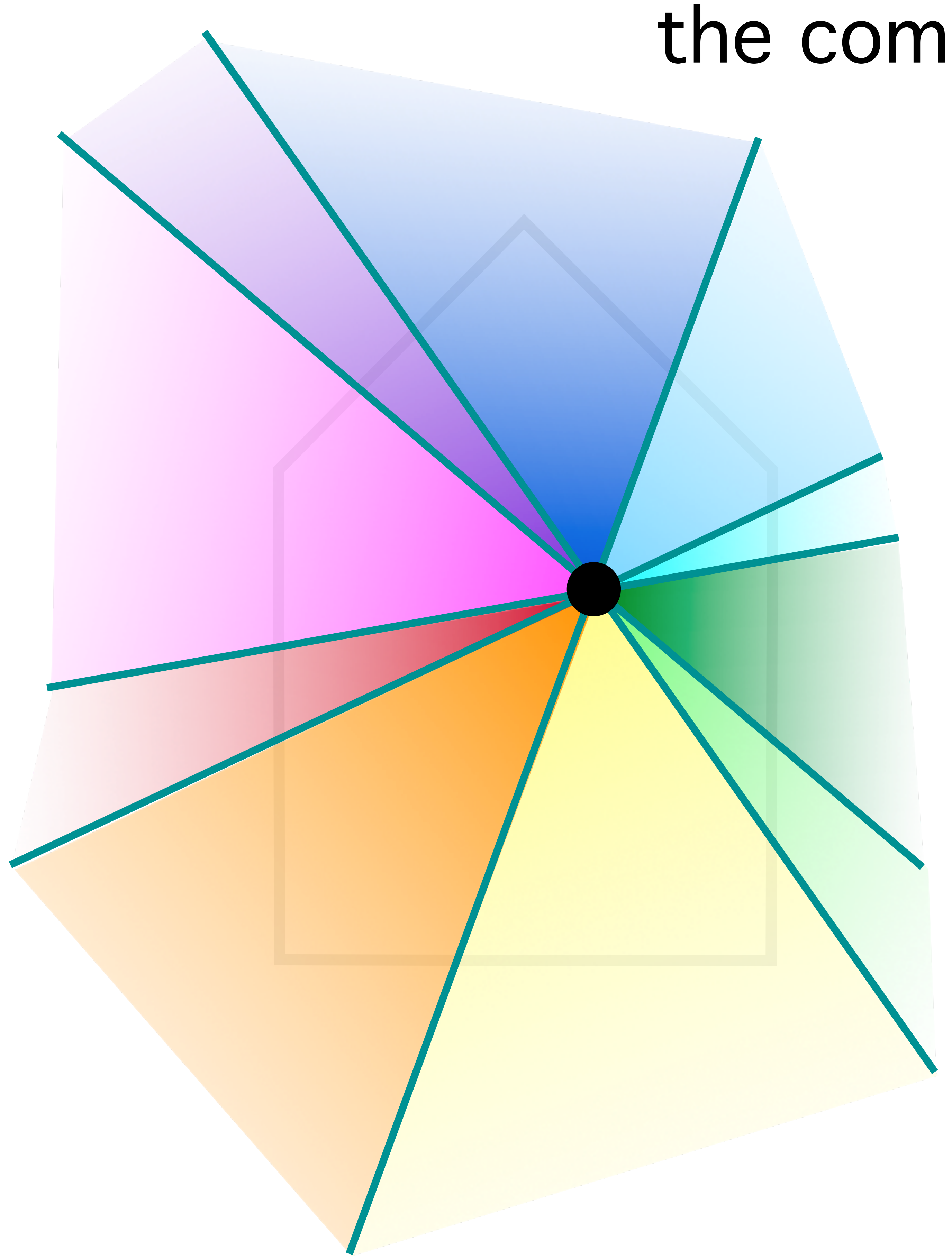




Analogously for all affine hyperplanes

$$\mathscr{H}_P = \{v^{\perp} \subset \mathbb{R}^{d+1} \mid v \in \text{vertices}(P), P \subset \mathbb{R}^d \times \{1\}\}$$

Hyperplane arrangements



In each cell of this hyperplane arrangement,
the combinatorial type of the slice does not change

Finitely many sets of slices

Rational functions $\frac{p(x)}{q(x)}$

Since the combinatorial type does not change,
we can get a nice parametrization of
the volume of the slices in each cell

Analogously for all affine hyperplanes

$$\mathscr{H}_P = \{v^\perp \subset \mathbb{R}^{d+1} \,|\, v \in \text{vertices}(P),\, P \subset \mathbb{R}^d \times \{1\}\}$$

Let's optimize!

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The best ways to slice a Polytope

by Marie-Charlotte Brandenburg, Jesús A. De Loera and Chiara Meroni;

Math. Comp. **94** (2025), 1003-1042

DOI: https://doi.org/10.1090/mcom/4006

Theorem [B,DL,M]: Let $P \subset \mathbb{R}^d$ be a polytope.
For fixed d, we can find the slice $P \cap H$ with

- $\max \operatorname{vol}(P \cap H)$
- $\min \operatorname{vol}(P \cap H)$ (through a fixed point)
- $\max \#\text{vertices}(P \cap H)$
- $\max \#\text{edges}(P \cap H)$
- ...

in polynomial time.

or projection $\pi_H(P)$,
or half-space $P \cap H^+$

Cannot hope for more, in general

Theorem [B,DL,M]:
It is (#P-)hard to compute the volume of the slice with largest volume.

Theory and practice

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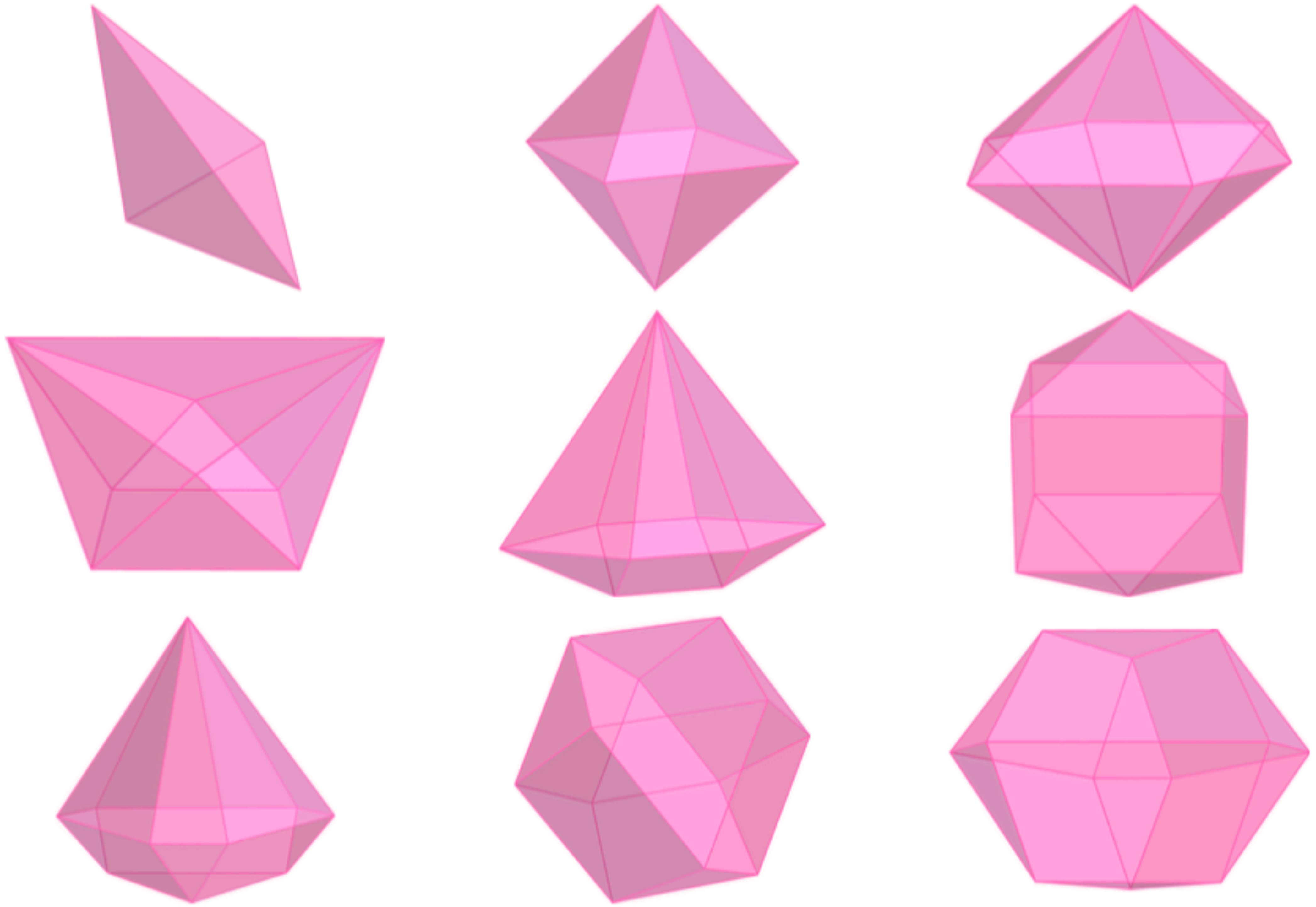
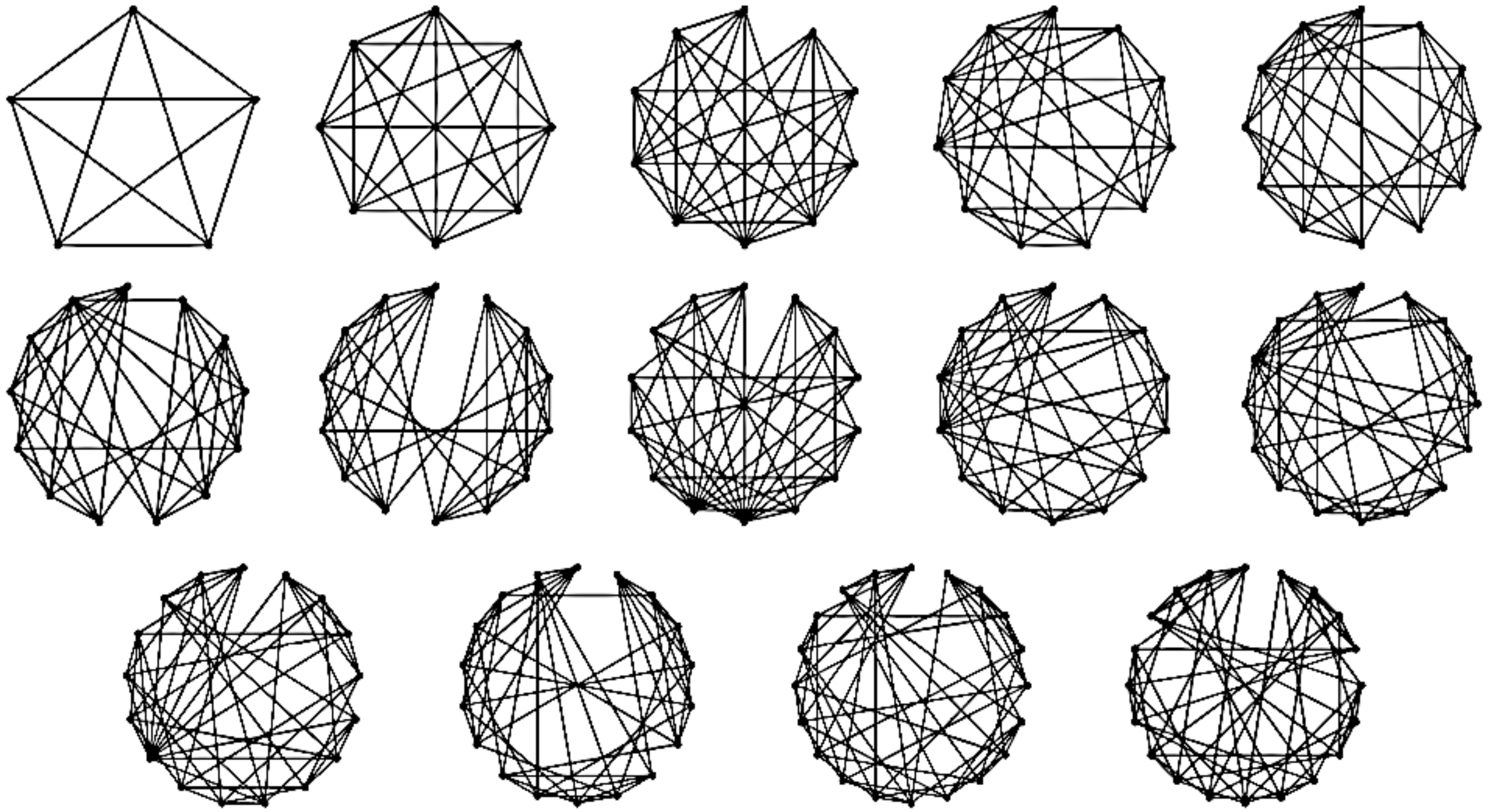
- SageMath
- Julia

https://mathrepo.mis.mpg.de/BestSlicePolytopes/

Platonic solids





Slices of the cube

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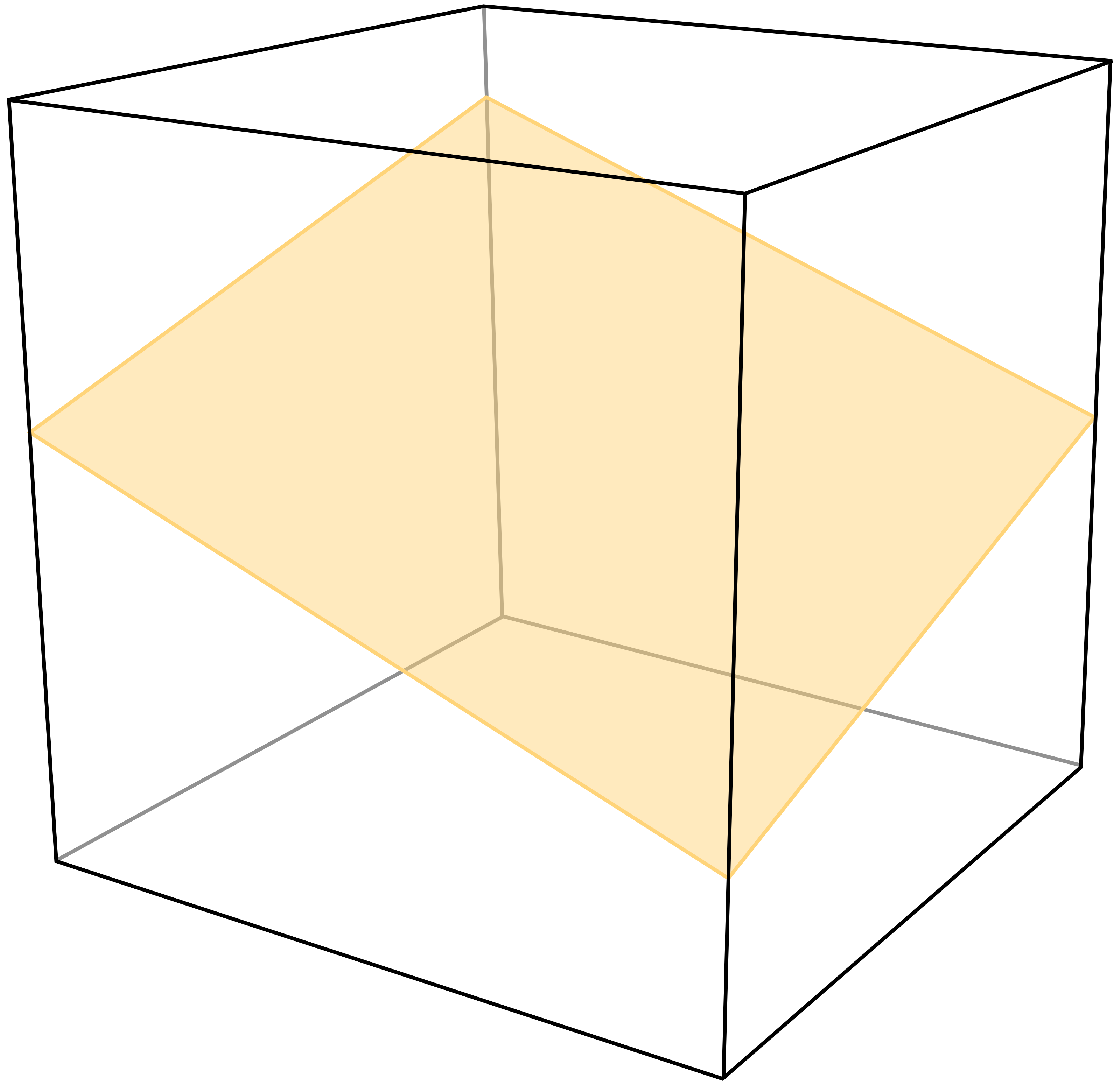


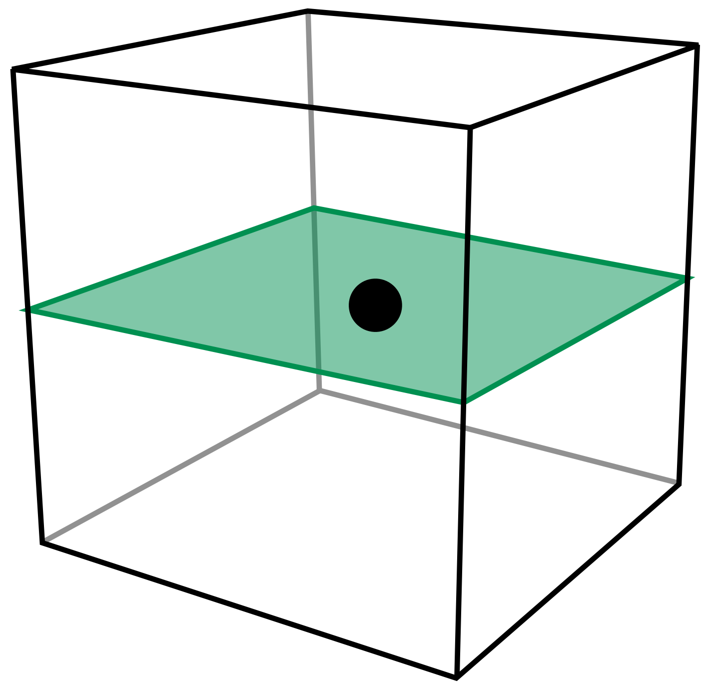
Metric: what is the minimal/maximal volume of slices of $[-1,1]^d$ through the origin?

Slicing the Cube in R^n and Probability (Bounds for the Measure of a Central Cube Slice in R^n by Probability Methods)

Douglas Hensley

Proceedings of the American Mathematical Society, Vol. 73, No. 1 (Jan., 1979), pp. 95-100 (6 pages)

minimal = parallel to a $(d-1)$-dim face

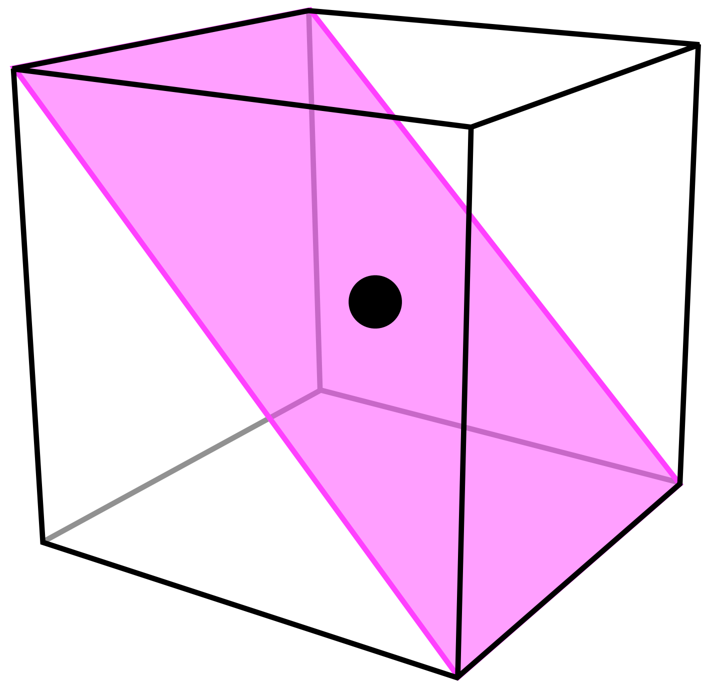


Cube Slicing in R^n

Keith Ball

Proceedings of the American Mathematical Society, Vol. 97, No. 3 (Jul., 1986), pp. 465-473 (9 pages)

maximal = containing a $(d-2)$-dim face



What about slices at distance t?

Open, very active area of research:
G. Ambrus, B. Gárgyán, H. König, V. Milman, L. Pournin, A. Zvavitch, ...

Slices of the cube

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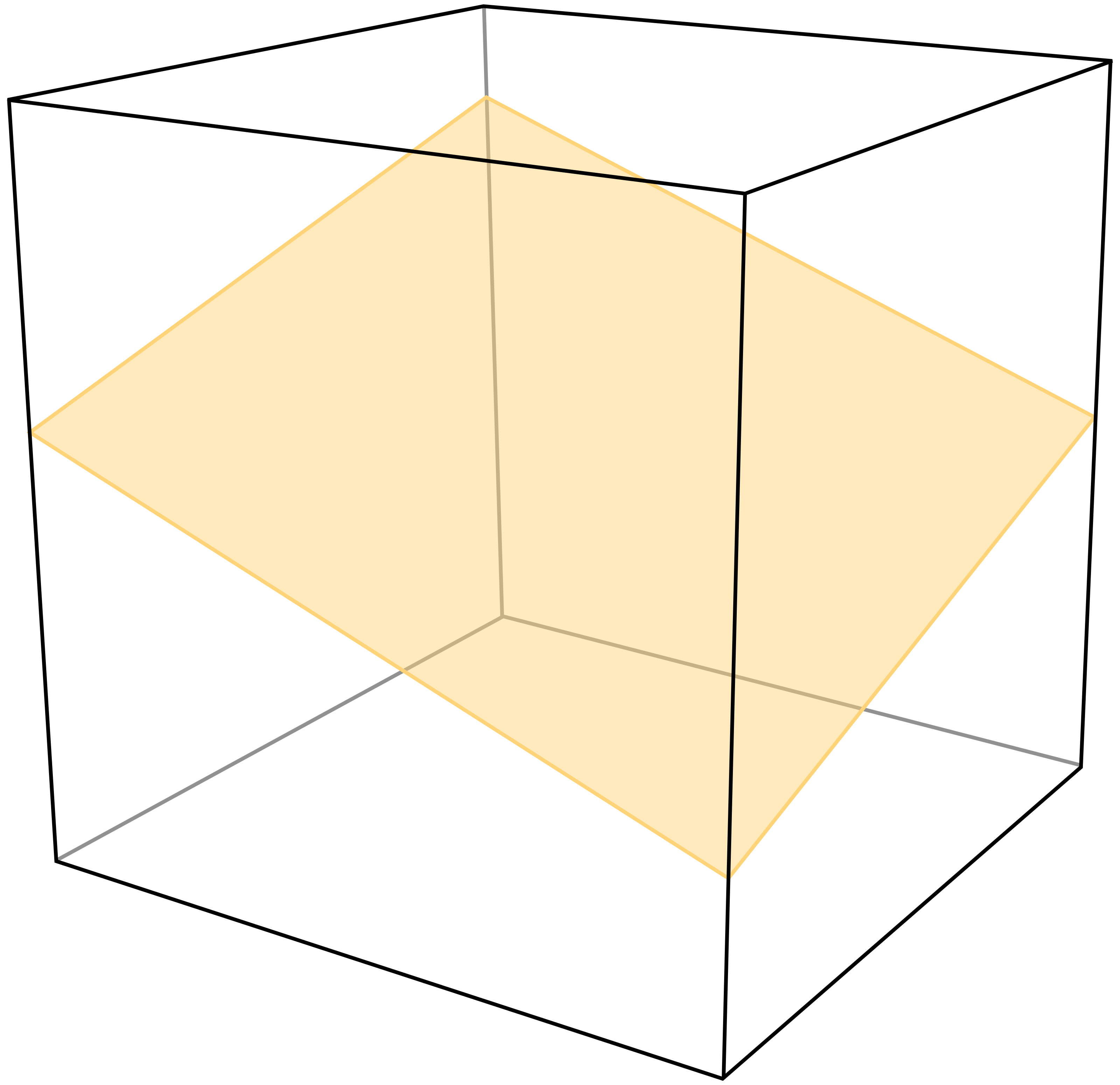
Combinatorial: what are the possible number of vertices of a slice of the cube?



arXiv > math > arXiv:2412.12419

On the Number of Vertices in a Hyperplane Section of a Polytope

Jesús A. De Loera, Gyivan Lopez-Campos, Antonio J. Torres

Dimension	SVS
2	$\{1,2\}$
3	$\{1,2,3,4,5,6\}$
4	$\{1,2,4,6,7,8,9,10,11,12\}$
5	$\{1,2,4,5,8,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30\}$
6	$\{1,2,4,6,8,10,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58,60\}$

Sequences of vertices from slices of hypercubes (SVS) up to dimension 6.

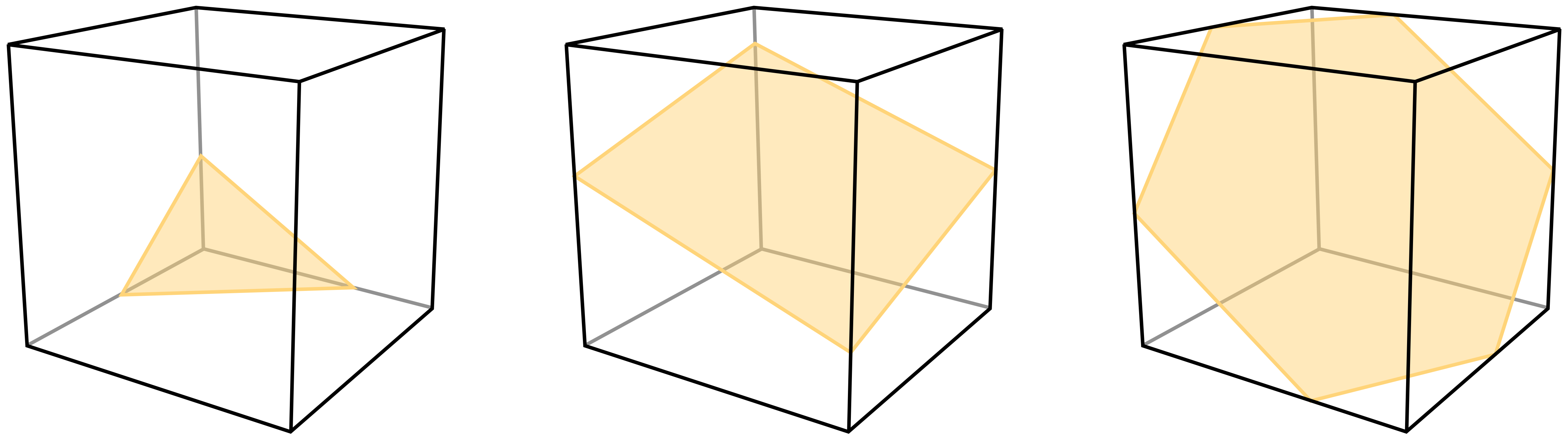
What about the possible combinatorial types?

C.M.

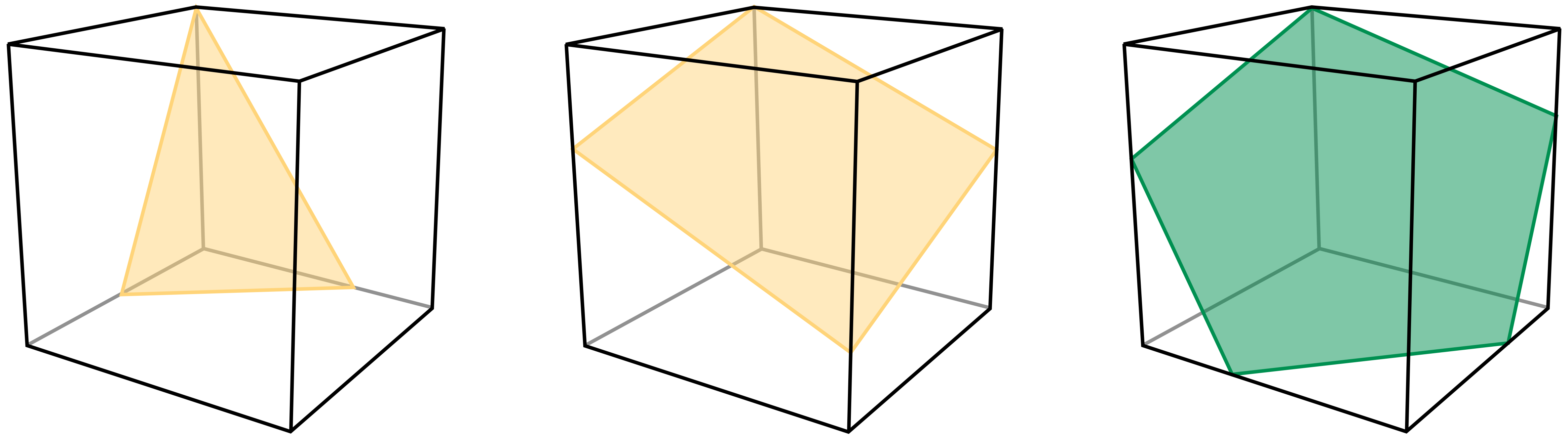
Marie-Charlotte Brandenburg

3-dimensional cube

Slices through zero vertices of the cube:



Slices through one vertex of the cube:



Slices through two vertices of the cube: triangle, quadrilateral (nothing new)

Slices through three vertices of the cube: triangle, quadrilateral (nothing new)
(lin. indep.)

4 combinatorial types

4-dimensional cube

ETH zürich

G. Nakamura, and A. Sasaki, Bessatu Suri-kagaku 10, Puzzle V, (1980).

non peer-reviewed Japanese magazine on math and related topics

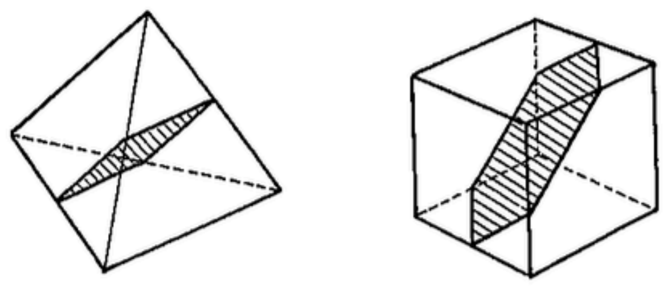
四次元の超立方体を切る

PUZZLE

中村　義作
佐々木　章夫

1.　まえがき

正四面体や立方体には，ちょっと気がつきにくい断面がある．正四面体にたいしては，図1(a)のように，これを4つの辺の中点を通る平面で切



(a)　(b)

図 1

ると，断面に正方形が現われる．また立方体にたいしては，図1(b)のように，6つの辺の中点を通る平面で切ると，断面に正六角形が現われる．これらのことは，パズルに詳しい読者には常識であるが，ふつうの人にはなかなか想像がつかないらしい．その証拠には，正四面体を図1(a)の2つの立体に切り分けたパズル玩具が，かなり以前から市販されている．

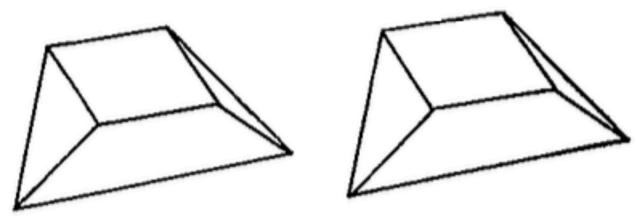


図 2

このパズルでは，正四面体を2つに切り分けた立体をさきに与えておく．この立体は，図2のように，ちょうど同じ形をしているところが面白い．問題は，「この2つの立体を組み合わせて，正四面体を作って下さい」というものであるが，いろいろの人に試してみると，なかなか作れないで苦心している．人間の盲点をついた面白いパズルである．まだ，やったことのない読者は，一度は誰かに試してみる価値がある．

ところで，正四面体や立方体のように，3次元の立体の断面を調べるときは，コンニャクのようなものを使えば，断面を実物で確認することができる．このため，手間さえ惜しまなければ，どんな立体の断面も求められるはずである．しかし，これが4次元の立体になると，少し話が面倒になる．なにしろ，われわれの住む世界が3次元のため，4次元の立体は想像上のものにすぎず，コンニャクのようなもので作るわけにはいかない．このため，想像をたくましくするか，なにかの数学に頼るしか方法がないが，そこには予想もつかないさまざまの断面が現われることになる．

この小論では，立方体を4次元に拡張した超立方体にたいして，いろいろの角度や深さから切ったときの断面を調べる．4次元の立体として超立方体を選んだのは，超立方体がわれわれの直感にもっとも結びやすいことと，断面の種類がちょうど手頃なためである．なお筆者の一人は，この超立方体をある一定の方向から平行に切り進めたときの断面の系列を紹介したことがあるが[1)]，この小論はその一般化であり，またすべての断面の完全な調査でもある．

2.　4次元の超立方体

まず，4次元の超立方体について，簡単な説明をしておこう．超立方体を考えるには，3次元のふつうの立方体を調べて，それを4次元に類推させるのがわかりやすい．平面上の正方形を，その平面と垂直な方向に1辺の長さだけ平行移動させると，3次元の立方体ができる．このことから，立方体に含まれる頂点と辺と平面の個数は，つぎのように計算される．まず頂点は，移動するまえの正方形の4頂点と，移動したあとの4頂点からなり，その合計は8個である．つぎに辺は，移動するまえの正方形の4辺と，移動したあとの4辺と，正方形の4頂点の移動からできる4辺からなり，その合計は12個である．最後に面は，移動するまえの正方形そのものの1面と，移動したあとの1面と，正方形の4辺の移動からできる4面からなり，その合計は6個である．もちろん，このような面倒な計算をしなくても，立方体の頂点と辺と面の個数は簡単に計算できるが，この方法が4次元の超立方体にも適用できるのである．

超立方体を作るには，3次元の立方体を，その立方体を含む空間と垂直な方向に1辺の長さだけ平行移動させればよい．ただし，われわれの住む3次元の世界では，立方体を含む空間と垂直な方向は存在しないから，想像上の話とする以外にない．しかし，この4次元の超立方体についても，それに含まれる頂点や辺や面の個数は，立方体と同じように計算できる．まず頂点は，移動するまえの8頂点と，移動したあとの8頂点からなり，その合計は16個である．つぎに辺は，移動するまえの立方体の12辺と，移動したあとの12辺と，立方体の8頂点の移動からできる8辺からなり，その合計は32個である．つぎに面は，移動するまえの立方体の6面と，移動したあとの6面と，立方体の12辺の移動からできる12面からなり，その合計は24個である．超立方体では，少し奇妙ではあるが，この他にその表面を覆っている何個かの立方体がある．その個数は，移動するまえの立方体そのものの1個と，移動したあとの1個と，立方体の6面の移動からできる6個の合計で，全部で8個である．

以上のことから，4次元の超立方体の表面は8

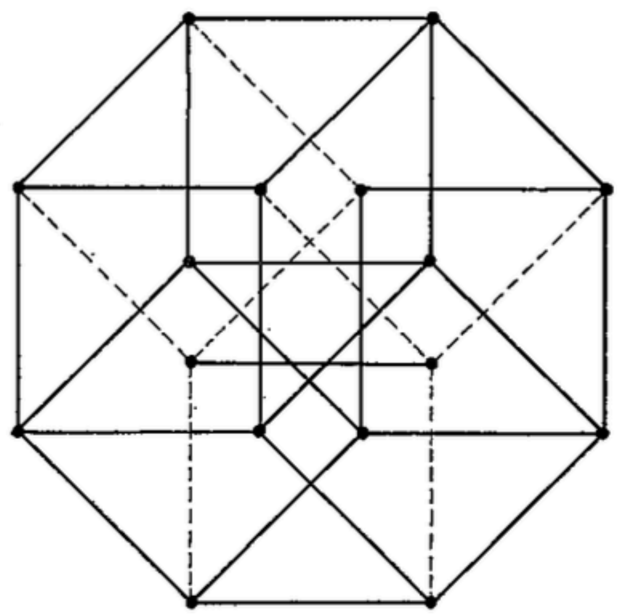


図 3

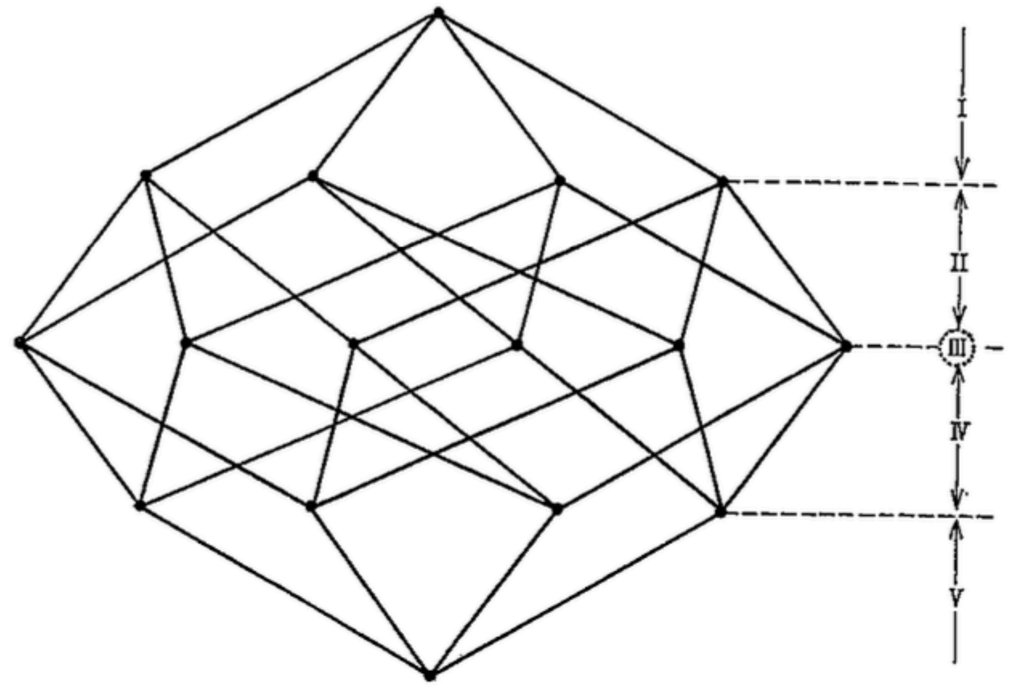


図 4

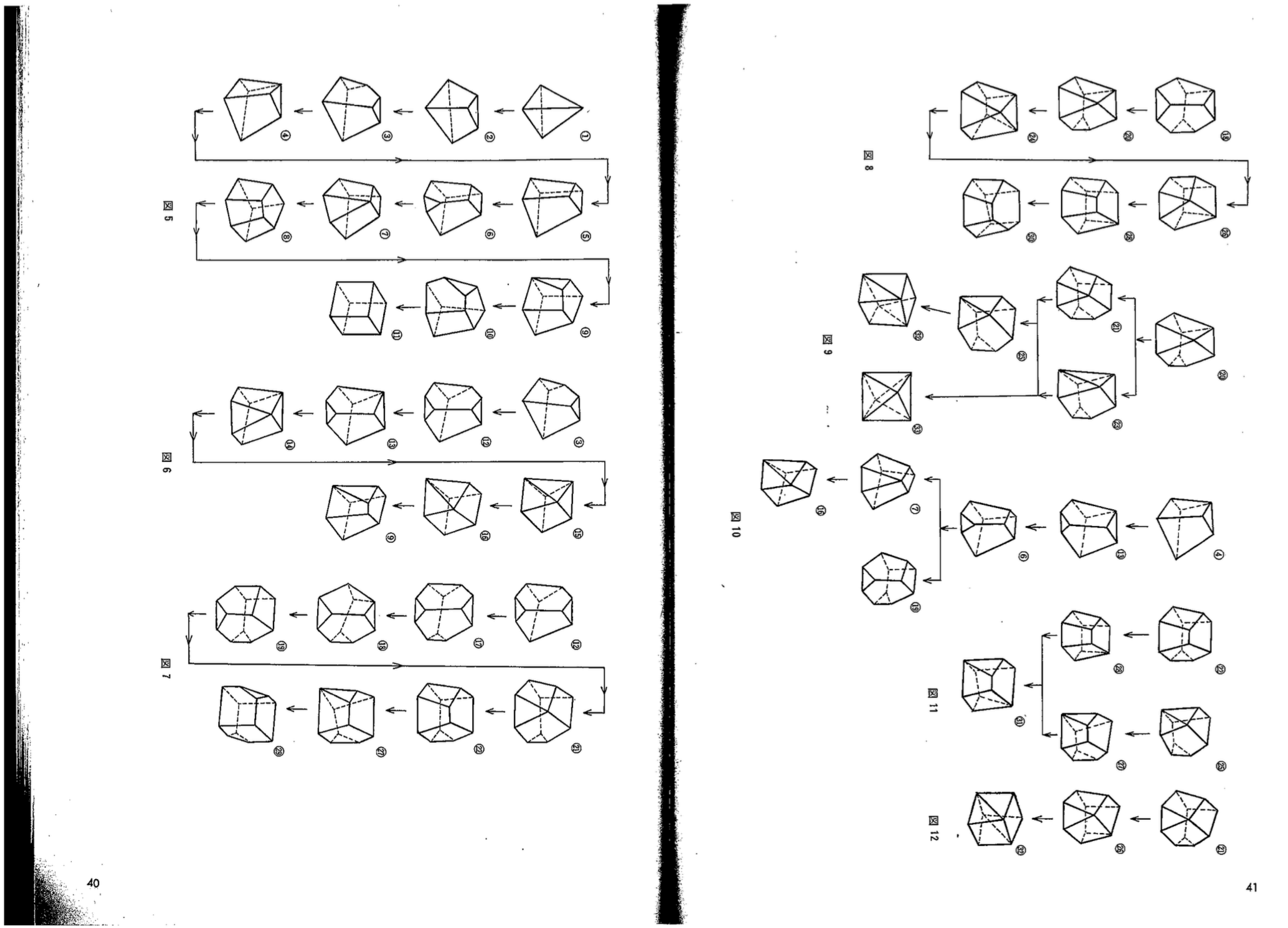
4-dimensional cube

ETH zürich

G. Nakamura, and A. Sasaki, Bessatu Suri-kagaku 10, Puzzle V, (1980).

non peer-reviewed Japanese magazine on math and related topics

30 combinatorial types



5-dimensional cube

Sections of Hyper-Cube in Five Dimensions

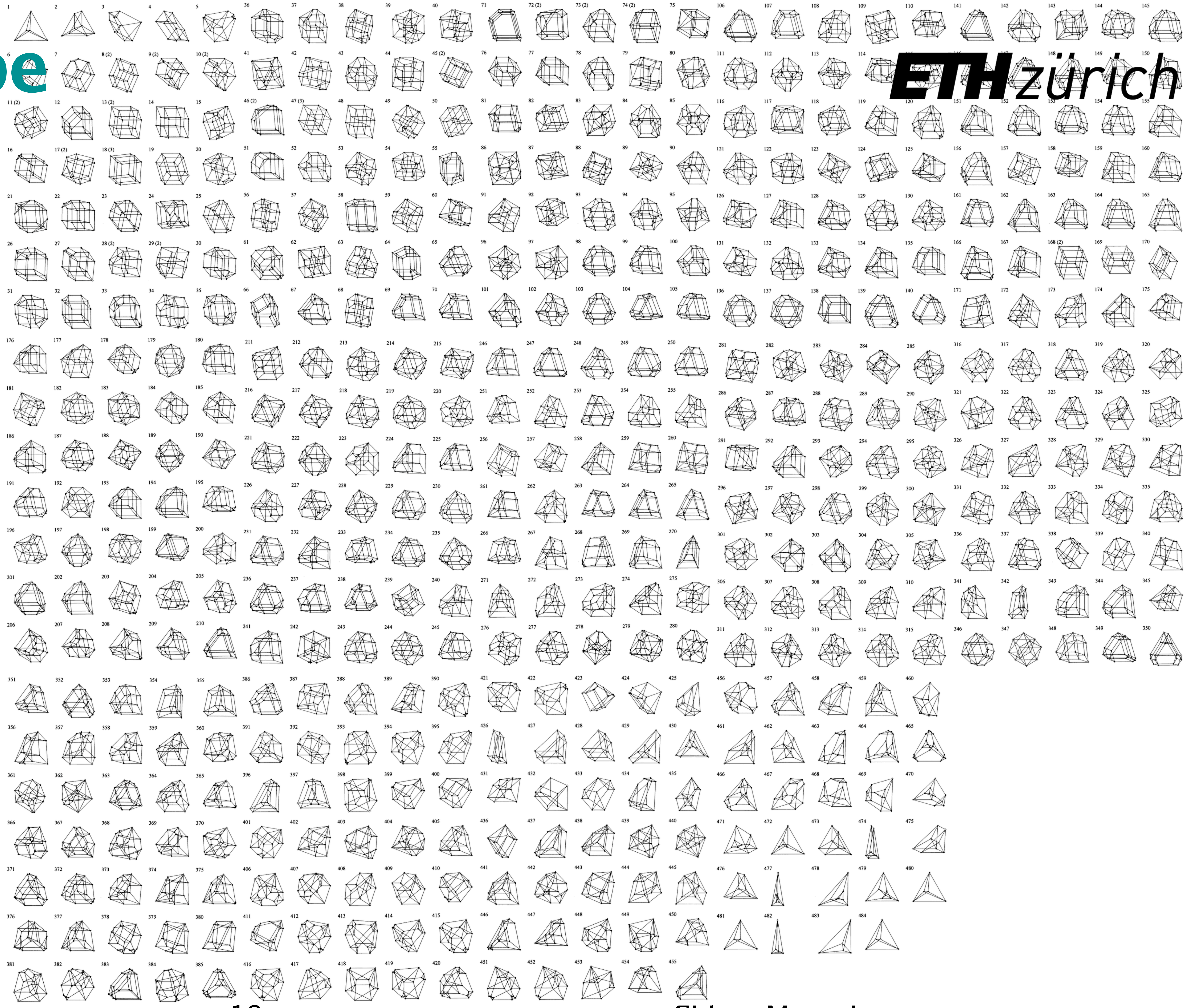
Hiroshi FUKUDA, Nobuaki MUTO, Kikuko GOTO

and

Gisaku NAKAMURA

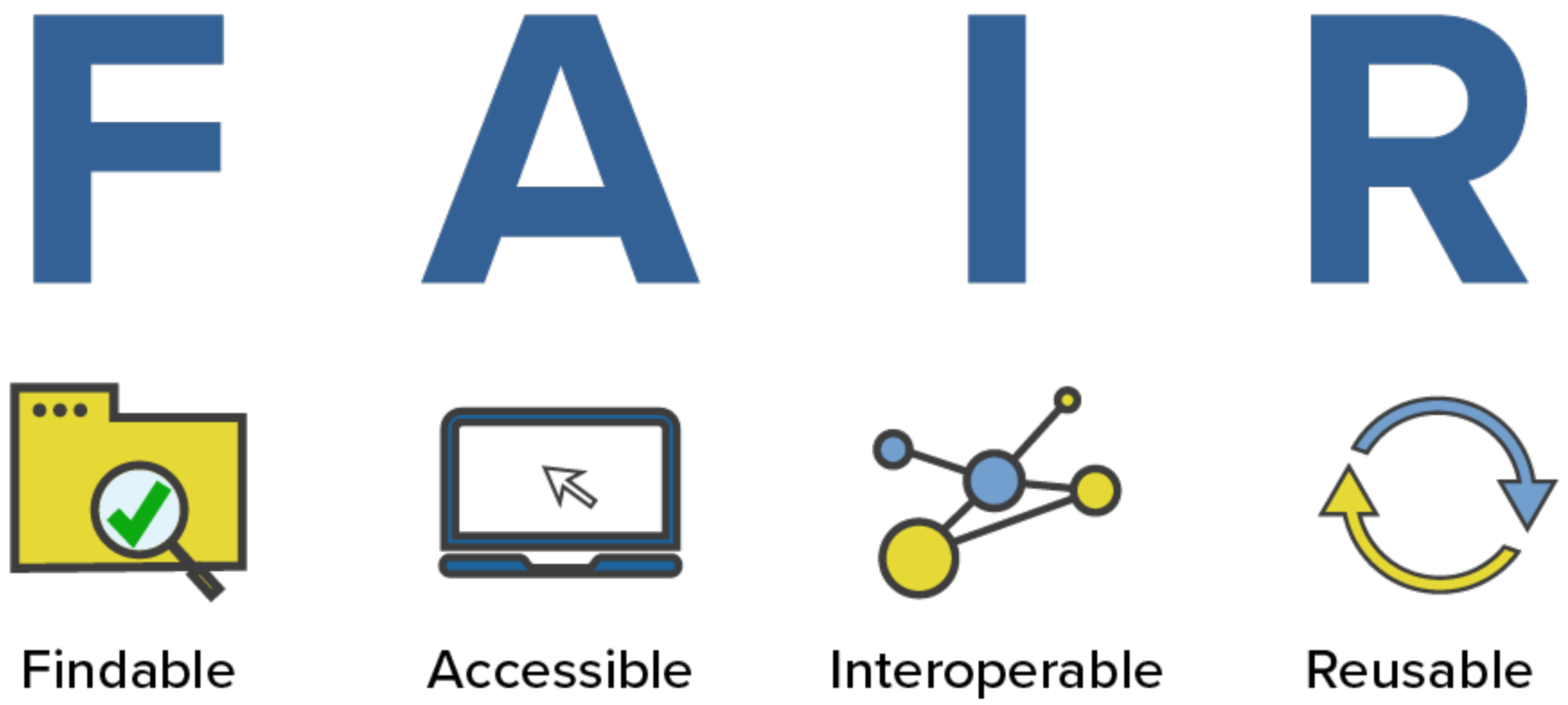
published in Forma, 1997
only preprint available online

484 "combinatorial types"



Our goals

Verify, prove, refine and provide open-source code for $d = 4,5$.

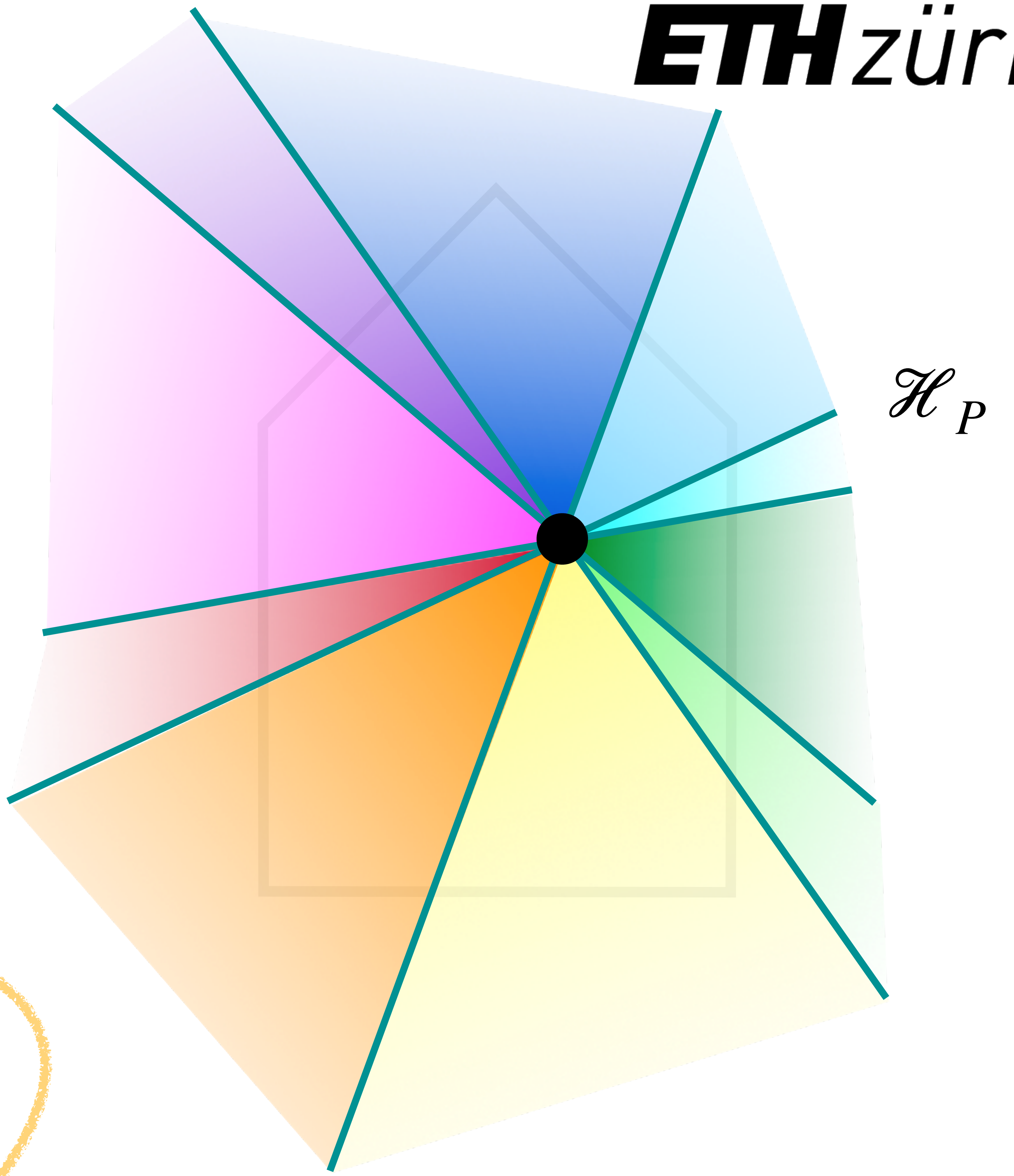


Compute $d = 6$.

Strategy

Ok for $d \leq 4$,
Out of reach for $d \geq 5$

- Define the hyperplane arrangement $\mathscr{H}_P$
- Compute all the cells, and the rays of each cell
- Find a point per cell with a certified numerical algorithm
- By summing the rays, find a point inside each cell
- Each such point defines a slice
- Compare the combinatorial types of all these slices



These are our candidates

For $d = 4$ there are 16486

Numerical Algebraic Geometry

ETH zürich

COMPOSITIO MATHEMATICA

The maximum likelihood degree of a very affine variety

June Huh

arXiv > cs > arXiv:2409.09622

Computing Arrangements of Hypersurfaces

Paul Breiding, Bernd Sturmfels, Kexin Wang

arXiv > math > arXiv:2412.20869

Hypersurface Arrangements with Generic Hypersurfaces Added

Bernhard Reinke, Kexin Wang

Hyperplane arrangement $\mathscr{H}_P$

$\downarrow$

Define a Morse function $f_{\mathscr{H}_P}$

$\downarrow$

Compute the critical points of $f_{\mathscr{H}_P}$ **HomotopyContinuation.jl**

$\downarrow$

One point in the interior of each cell of $\mathscr{H}_P$

Numerical Algebraic Geometry

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Compute the critical points of $f_{\mathcal{H}_P}$

system of polynomial equations

$$F(x) = \begin{cases} f_1(x) = 0 \\ \vdots \\ f_d(x) = 0 \end{cases} \qquad x \in \mathbb{R}^d$$

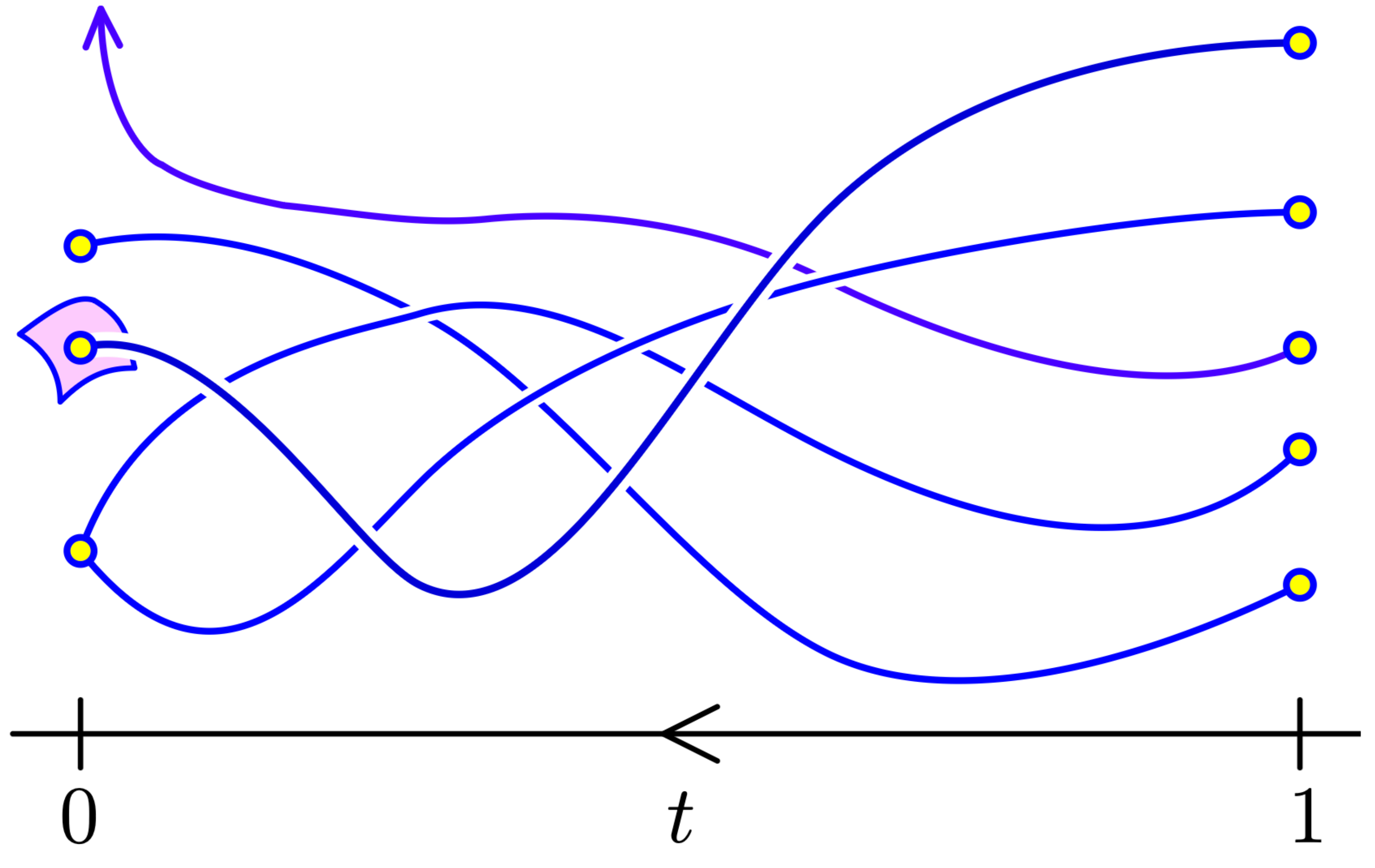
Idea: solve an easy system $G(x) = \begin{cases} g_1(x) = 0 \\ \vdots \\ g_d(x) = 0 \end{cases}$

then track the solutions of G to
those of F using a homotopy

HomotopyContinuation.jl

NUMERICAL NONLINEAR ALGEBRA

DANIEL J. BATES, PAUL BREIDING, TIANRAN CHEN, JONATHAN D. HAUENSTEIN, ANTON LEYKIN, AND FRANK SOTTILE



$F(x) = 0$ $\qquad$ $G(x) = 0$

Fast (numerical methods) but then certifies the solutions!

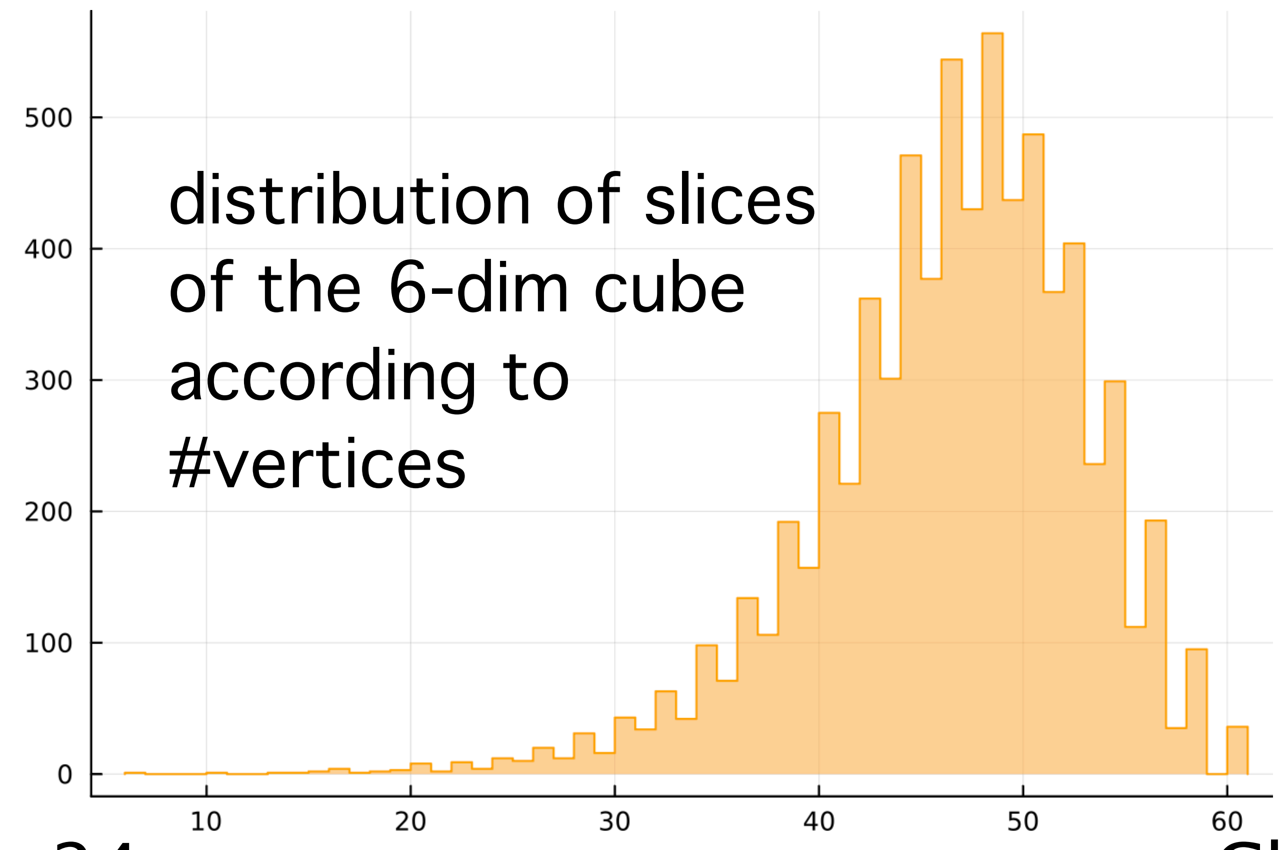
Give me the numbers!

these are double checked with the exact algorithm

ETH zürich

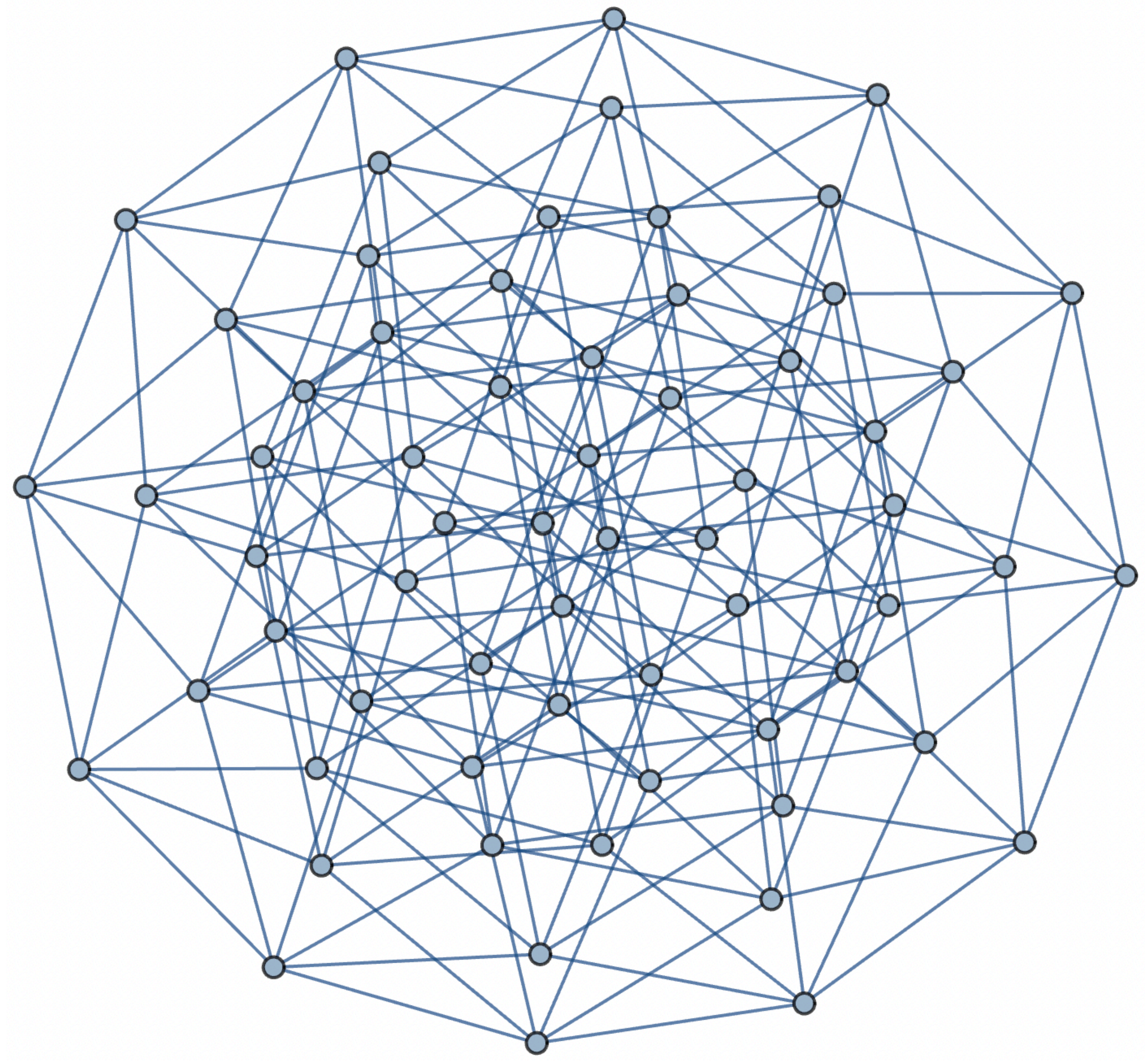
# lin. indep. vertices of $[-1,1]^d$ in the slice	0	1	2	3	4	5	6	TOT
$d = 4$	12	14 (7 new)	14 (6 new)	10 (4 new)	6 (1 new)			30
$d = 5$	58	103 (81 new)	129 (96 new)	105 (73 new)	52 (31 new)	14 (5 new)		344
$d = 6$	554	1482 (1276 new)	2296 (2078 new)	2179 (1917 new)	1261 (1053 new)	413 (312 new)	60 (26 new)	7326

Out of 15.028.134 candidates



What more?

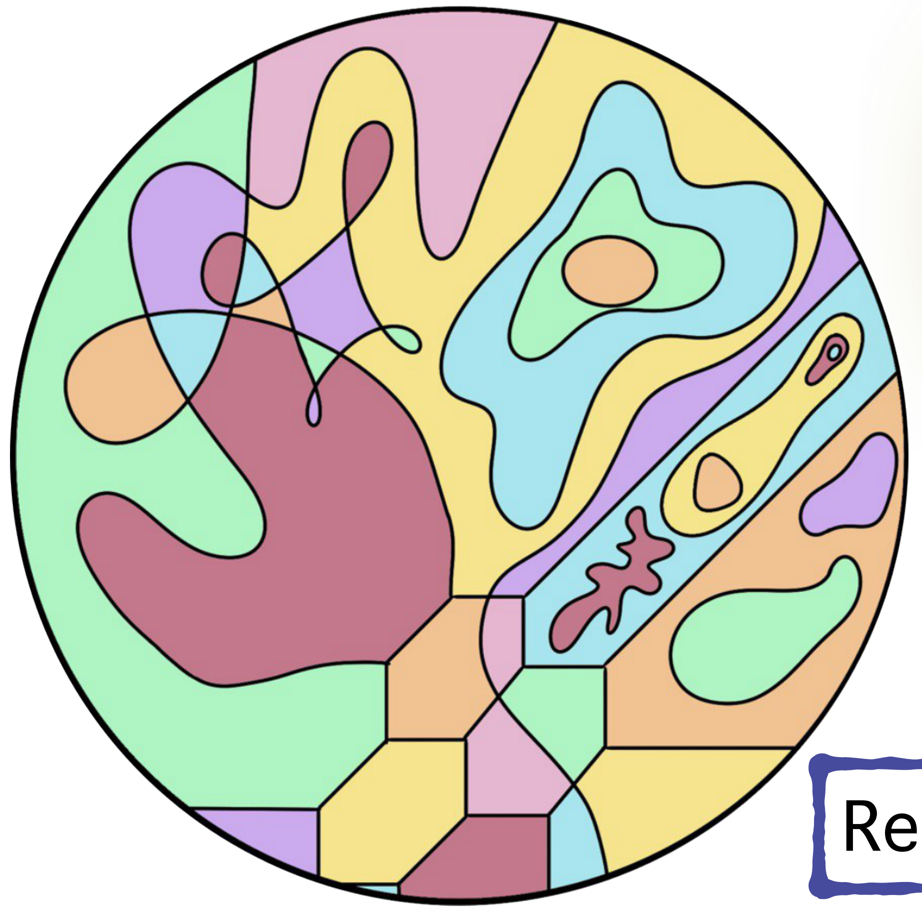
ETH zürich



- Reproduce the Japanese papers, and their “combinatorial types”
- Exploit symmetry as much as possible
- Create database with all slices, f-vectors, ...

Computations in Algebraic Geometry:
Complex, Real, and Tropical

Workshop from 25 to 29 August 2025



Come and present a poster!

Co-organized with Simon Telen

Registration deadline: May 15

- Combinatorial version of Busemann-Petty and Bourgain
- Lattice points version of Busemann-Petty and Bourgain
- Lower dimensional slices
- Computational complexity statements
- Probabilistic statements
- ...

Thank you!