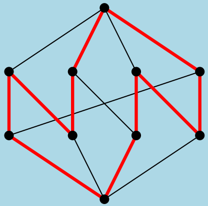
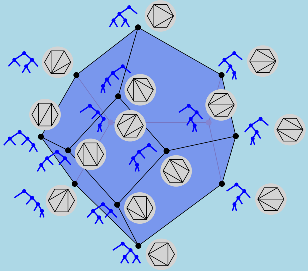
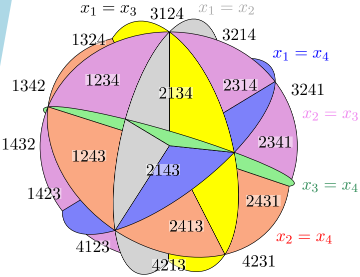
Hamiltonicity in polytopes and hyperplane arrangements

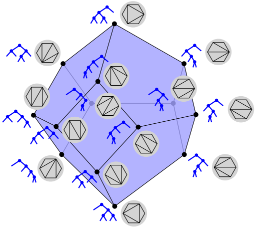
Sofia Brenner

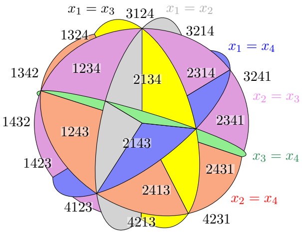
Charles University (KAM)







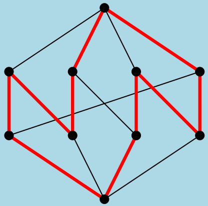




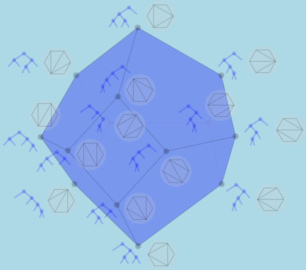
Listing faces of polytopes
with Nastaran Behrooznia, Arturo Merino, Torsten Mütze, Christian Rieck, and Francesco Verciani

Traversing regions of hyperplane arrangements and their lattice quotients
with Jean Cardinal, Thomas McConville, Arturo Merino, and Torsten Mütze

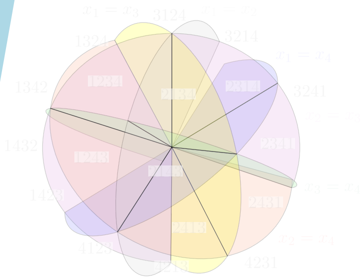
Introduction



Face Lattices



Regions in Hyperplane Arrangements



Hamiltonian cycles

Hamiltonian cycles

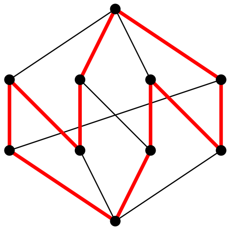
Hamiltonian cycles and paths

A Hamiltonian cycle (path) in a graph G is a cycle (path) that visits every vertex precisely once.

Hamiltonian cycles

Hamiltonian cycles and paths

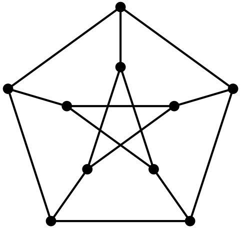
A Hamiltonian cycle (path) in a graph G is a cycle (path) that visits every vertex precisely once.



Hamiltonian cycles

Hamiltonian cycles and paths

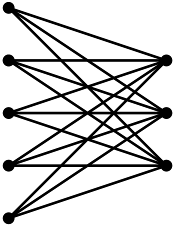
A Hamiltonian cycle (path) in a graph G is a cycle (path) that visits every vertex precisely once.



Hamiltonian cycles

Hamiltonian cycles and paths

A Hamiltonian cycle (path) in a graph G is a cycle (path) that visits every vertex precisely once.



Hamiltonian cycles

Hamiltonian cycles and paths

A Hamiltonian cycle (path) in a graph G is a cycle (path) that visits every vertex precisely once.

- NP-hard problem

Hamiltonian cycles

Hamiltonian cycles and paths

A Hamiltonian cycle (path) in a graph G is a cycle (path) that visits every vertex precisely once.

- NP-hard problem
- sufficient conditions (Dirac's theorem)

Hamiltonian cycles

Hamiltonian cycles and paths

A Hamiltonian cycle (path) in a graph G is a cycle (path) that visits every vertex precisely once.

- NP-hard problem
- sufficient conditions (Dirac's theorem)

Lovasz Conjecture

Every vertex-transitive graph has a Hamiltonian path.

Combinatorial generation

Combinatorial generation

Aim: listing of all combinatorial objects of a certain type, allowing only small modifications (flips) between consecutive elements ("Gray code")

Combinatorial generation

Aim: listing of all combinatorial objects of a certain type, allowing only small modifications (flips) between consecutive elements ("Gray code")

Example

Combinatorial generation

Aim: listing of all combinatorial objects of a certain type, allowing only small modifications (flips) between consecutive elements ("Gray code")

Example

- bit strings of length n, by bit flips

Combinatorial generation

Aim: listing of all combinatorial objects of a certain type, allowing only small modifications (flips) between consecutive elements ("Gray code")

Example

- bit strings of length n, by bit flips
- permutations on n symbols, by adjacent transpositions

Combinatorial generation

Aim: listing of all combinatorial objects of a certain type, allowing only small modifications (flips) between consecutive elements ("Gray code")

Example

- bit strings of length n, by bit flips
- permutations on n symbols, by adjacent transpositions
- noncrossing perfect matchings on a convex point set of size $4n$, by alternating 4-cycles

Combinatorial generation

Aim: listing of all combinatorial objects of a certain type, allowing only small modifications (flips) between consecutive elements ("Gray code")

Example

- bit strings of length n, by bit flips
- permutations on n symbols, by adjacent transpositions
- noncrossing perfect matchings on a convex point set of size $4n$, by alternating 4-cycles

Flip graph

Combinatorial generation

Aim: listing of all combinatorial objects of a certain type, allowing only small modifications (flips) between consecutive elements ("Gray code")

Example

- bit strings of length n, by bit flips
- permutations on n symbols, by adjacent transpositions
- noncrossing perfect matchings on a convex point set of size $4n$, by alternating 4-cycles

Flip graph

- vertices $=$ objects

Combinatorial generation

Aim: listing of all combinatorial objects of a certain type, allowing only small modifications (flips) between consecutive elements ("Gray code")

Example

- bit strings of length n, by bit flips
- permutations on n symbols, by adjacent transpositions
- noncrossing perfect matchings on a convex point set of size $4n$, by alternating 4-cycles

Flip graph

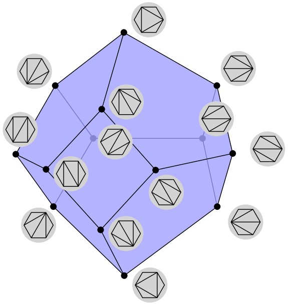
- vertices = objects
- edge = feasible flip

Combinatorial generation

Aim: listing of all combinatorial objects of a certain type, allowing only small modifications (flips) between consecutive elements ("Gray code")

Example

- bit strings of length n, by bit flips
- permutations on n symbols, by adjacent transpositions
- noncrossing perfect matchings on a convex point set of size $4n$, by alternating 4-cycles



Flip graph

- vertices = objects
- edge = feasible flip

Combinatorial generation

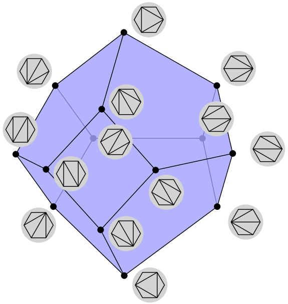
Aim: listing of all combinatorial objects of a certain type, allowing only small modifications (flips) between consecutive elements ("Gray code")

Example

- bit strings of length n, by bit flips
- permutations on n symbols, by adjacent transpositions
- noncrossing perfect matchings on a convex point set of size $4n$, by alternating 4-cycles

Flip graph

- vertices = objects
- edge = feasible flip
- combinatorial Gray code = Hamiltonian path



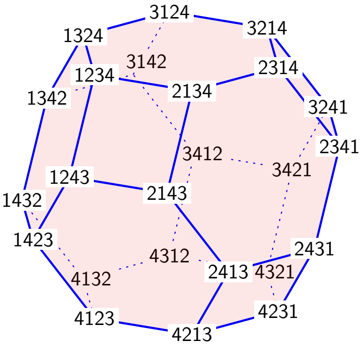
Steinhaus-Johnson-Trotter Algorithm

Steinhaus-Johnson-Trotter Algorithm

Aim: Gray code for all permutations on n symbols by adjacent transpositions

Steinhaus-Johnson-Trotter Algorithm

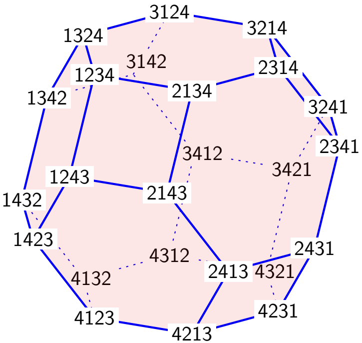
Aim: Gray code for all permutations on n symbols by adjacent transpositions



Steinhaus-Johnson-Trotter Algorithm

Aim: Gray code for all permutations on n symbols by adjacent transpositions

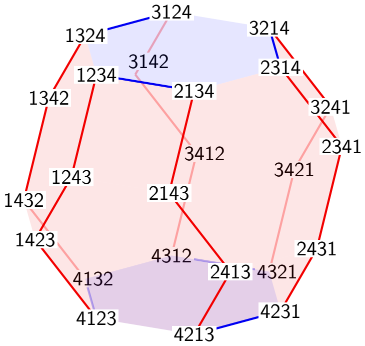
n	
1	1
2	12, 21
3	123, 132, 312, 321, 231, 213
4	1234, 1243, 1423, 4123, 4132, 1432, 1342, 1324, 3124, 3142, 3412, 4312, 4321, 3421, 3241, 3214, 2314, 2341, 2431, 4231, 4213, 2413, 2143, 2134



Steinhaus-Johnson-Trotter Algorithm

Aim: Gray code for all permutations on n symbols by adjacent transpositions

n	
1	1
2	12, 21
3	123, 132, 312, 321, 231, 213
4	1234, 1243, 1423, 4123, 4132, 1432, 1342, 1324, 3124, 3142, 3412, 4312, 4321, 3421, 3241, 3214, 2314, 2341, 2431, 4231, 4213, 2413, 2143, 2134



Zigzag framework

Zigzag framework

Aim: Gray codes for objects that are in bijection with certain subsets of permutations ("zigzag languages")

Zigzag framework

Aim: Gray codes for objects that are in bijection with certain subsets of permutations ("zigzag languages")

- binary reflected Gray code [Gray, 1953]

Zigzag framework

Aim: Gray codes for objects that are in bijection with certain subsets of permutations ("zigzag languages")

- binary reflected Gray code [Gray, 1953]
- Steinhaus-Johnson-Trotter algorithm [Steinhaus, Johnson, Trotter, 1960s]

Zigzag framework

Aim: Gray codes for objects that are in bijection with certain subsets of permutations ("zigzag languages")

- binary reflected Gray code [Gray, 1953]
- Steinhaus-Johnson-Trotter algorithm [Steinhaus, Johnson, Trotter, 1960s]
- pattern avoiding permutations [Hartung, Hoang, Mütze, Williams, 2020]

Zigzag framework

Aim: Gray codes for objects that are in bijection with certain subsets of permutations ("zigzag languages")

- binary reflected Gray code [Gray, 1953]
- Steinhaus-Johnson-Trotter algorithm [Steinhaus, Johnson, Trotter, 1960s]
- pattern avoiding permutations [Hartung, Hoang, Mütze, Williams, 2020]
- lattice quotients of the weak order (quotientopes) [Hoang, Mütze, 2021] [Pilaud, Santos, 2019]

Zigzag framework

Aim: Gray codes for objects that are in bijection with certain subsets of permutations ("zigzag languages")

- binary reflected Gray code [Gray, 1953]
- Steinhaus-Johnson-Trotter algorithm [Steinhaus, Johnson, Trotter, 1960s]
- pattern avoiding permutations [Hartung, Hoang, Mütze, Williams, 2020]
- lattice quotients of the weak order (quotientopes) [Hoang, Mütze, 2021] [Pilaud, Santos, 2019]
- elimination trees of chordal graphs (graph associahedra of chordal graphs) [Cardinal, Merino, Mütze, 2022]

Zigzag framework

Aim: Gray codes for objects that are in bijection with certain subsets of permutations ("zigzag languages")

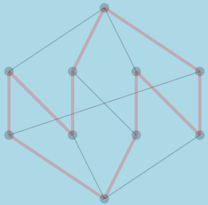
- binary reflected Gray code [Gray, 1953]
- Steinhaus-Johnson-Trotter algorithm [Steinhaus, Johnson, Trotter, 1960s]
- pattern avoiding permutations [Hartung, Hoang, Mütze, Williams, 2020]
- lattice quotients of the weak order (quotientopes) [Hoang, Mütze, 2021] [Pilaud, Santos, 2019]
- elimination trees of chordal graphs (graph associahedra of chordal graphs) [Cardinal, Merino, Mütze, 2022]
- (patter-avoiding) rectangulations (rectangulotopes) [Merino, Mütze, 2023] [Cardinal, Pilaud, 2025]

Zigzag framework

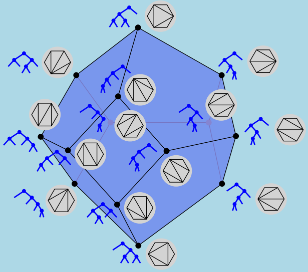
Aim: Gray codes for objects that are in bijection with certain subsets of permutations ("zigzag languages")

- binary reflected Gray code [Gray, 1953]
- Steinhaus-Johnson-Trotter algorithm [Steinhaus, Johnson, Trotter, 1960s]
- pattern avoiding permutations [Hartung, Hoang, Mütze, Williams, 2020]
- lattice quotients of the weak order (quotientopes) [Hoang, Mütze, 2021] [Pilaud, Santos, 2019]
- elimination trees of chordal graphs (graph associahedra of chordal graphs) [Cardinal, Merino, Mütze, 2022]
- (patter-avoiding) rectangulations (rectangulotopes) [Merino, Mütze, 2023] [Cardinal, Pilaud, 2025]
- ...

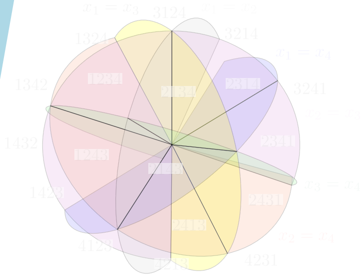
Introduction



Face Lattices



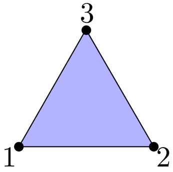
Regions in Hyper-
plane Arrangements

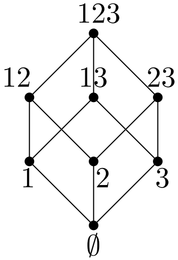


Face lattices of polytopes

Face lattices of polytopes

Given a polytope P, we consider (the cover graph of) its face lattice $L(P)$.

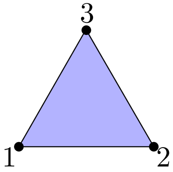


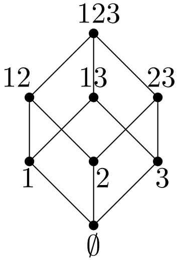


Face lattices of polytopes

Given a polytope P, we consider (the cover graph of) its face lattice $L(P)$.

- vertices are the faces of P (including $\emptyset$ and P)

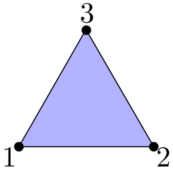


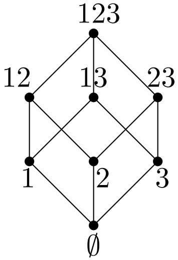


Face lattices of polytopes

Given a polytope P, we consider (the cover graph of) its face lattice $L(P)$.

- vertices are the faces of P (including $\emptyset$ and P)
- F_1 and F_2 are adjacent if $F_1 \subseteq F_2$ and $\dim F_2 = \dim F_1 + 1$





Hamiltonicity of face lattices

Hamiltonicity of face lattices

Question

For which polytopes P is the cover graph of $L(P)$ Hamiltonian?

Hamiltonicity of face lattices

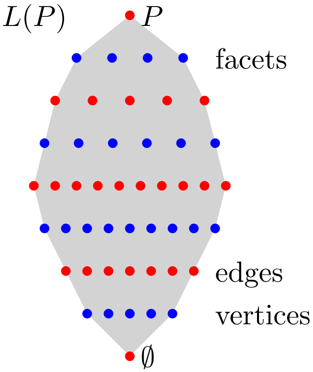
Question

For which polytopes P is the cover graph of $L(P)$ Hamiltonian?

Observation

The graph of $L(P)$ is bipartite and balanced.

Hamiltonicity of face lattices



Question

For which polytopes P is the cover graph of $L(P)$ Hamiltonian?

Observation

The graph of $L(P)$ is bipartite and balanced.

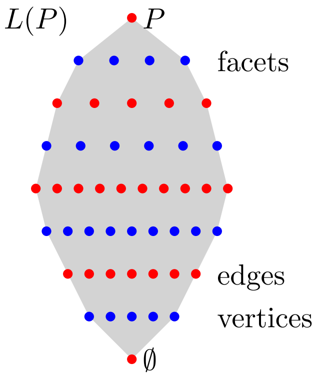
Hamiltonicity of face lattices

Conjecture

For any polytope P, the cover graph of $L(P)$ is Hamiltonian.

Observation

The graph of $L(P)$ is bipartite and balanced.



Some evidence

Some evidence

Theorem (Behrooznia, B., Merino, Mütze, Rieck, Verciani, 2026)

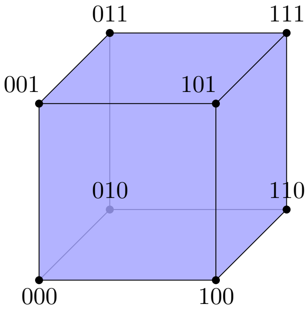
For the following classes of polytopes P, the cover graph of $L(P)$ has a Hamiltonian cycle:

Some evidence

Theorem (Behrooznia, B., Merino, Mütze, Rieck, Verciani, 2026)

For the following classes of polytopes P, the cover graph of $L(P)$ has a Hamiltonian cycle:

- hypercubes

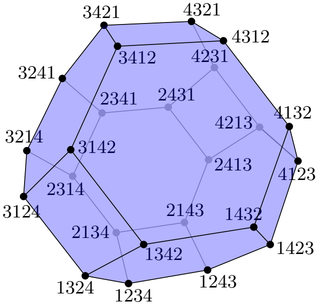


Some evidence

Theorem (Behrooznia, B., Merino, Mütze, Rieck, Verciani, 2026)

For the following classes of polytopes P, the cover graph of $L(P)$ has a Hamiltonian cycle:

- hypercubes
- permutahedra

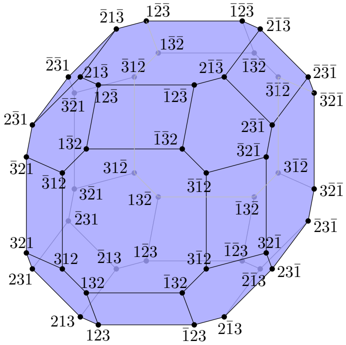


Some evidence

Theorem (Behrooznia, B., Merino, Mütze, Rieck, Verciani, 2026)

For the following classes of polytopes P, the cover graph of $L(P)$ has a Hamiltonian cycle:

- hypercubes
- permutahedra
- B-permutahedra



Some evidence

Theorem (Behrooznia, B., Merino, Mütze, Rieck, Verciani, 2026)

For the following classes of polytopes P, the cover graph of $L(P)$ has a Hamiltonian cycle:

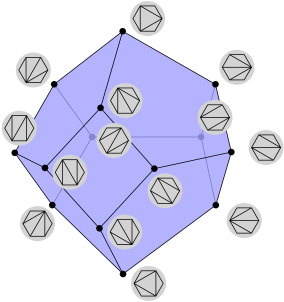
- hypercubes
- permutahedra
- B-permutahedra
- quotientopes

Some evidence

Theorem (Behrooznia, B., Merino, Mütze, Rieck, Verciani, 2026)

For the following classes of polytopes P, the cover graph of $L(P)$ has a Hamiltonian cycle:

- hypercubes
- permutahedra
- B-permutahedra
- quotientopes
- associahedra



Some evidence

Theorem (Behrooznia, B., Merino, Mütze, Rieck, Verciani, 2026)

For the following classes of polytopes P, the cover graph of $L(P)$ has a Hamiltonian cycle:

- hypercubes
- permutahedra
- B-permutahedra
- quotientopes
- associahedra
- graph associahedra of chordal graphs

Some evidence

Theorem (Behrooznia, B., Merino, Mütze, Rieck, Verciani, 2026)

For the following classes of polytopes P, the cover graph of $L(P)$ has a Hamiltonian cycle:

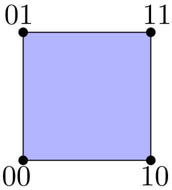
- hypercubes
- permutahedra
- B-permutahedra
- quotientopes
- associahedra
- graph associahedra of chordal graphs
- 3-dimensional polytopes

Example: Hypercubes

Example: Hypercubes

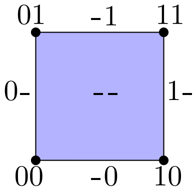
Theorem

The graph of $L(Q_n)$ admits a Hamiltonian cycle.



Example: Hypercubes

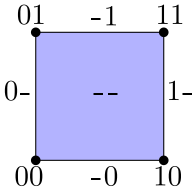
Theorem
The graph of $L(Q_n)$ admits a Hamiltonian cycle.

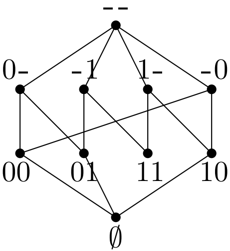


Example: Hypercubes

Theorem

The graph of $L(Q_n)$ admits a Hamiltonian cycle.

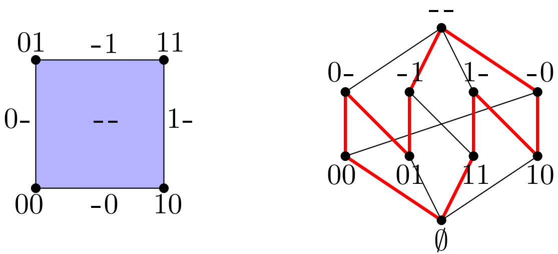




Example: Hypercubes

Theorem

The graph of $L(Q_n)$ admits a Hamiltonian cycle.



Example: Hypercubes

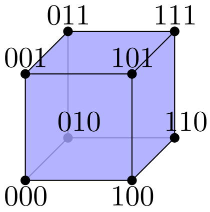
Theorem

The graph of $L(Q_n)$ admits a Hamiltonian cycle.

Example: Hypercubes

Theorem

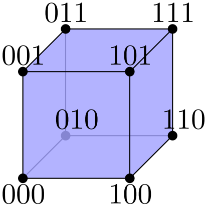
The graph of $L(Q_n)$ admits a Hamiltonian cycle.

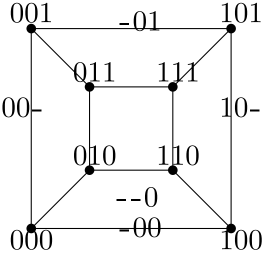


Example: Hypercubes

Theorem

The graph of $L(Q_n)$ admits a Hamiltonian cycle.

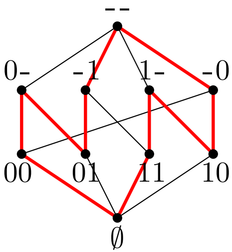


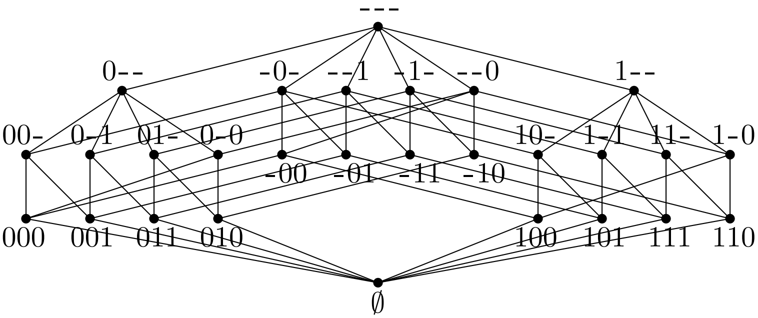


Example: Hypercubes

Theorem

The graph of $L(Q_n)$ admits a Hamiltonian cycle.

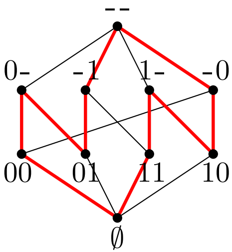


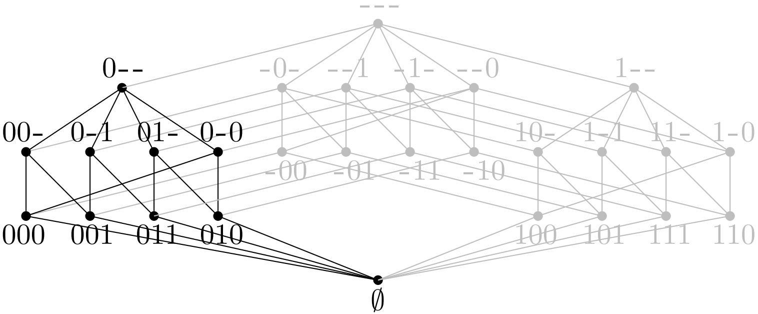


Example: Hypercubes

Theorem

The graph of $L(Q_n)$ admits a Hamiltonian cycle.

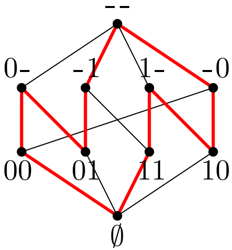


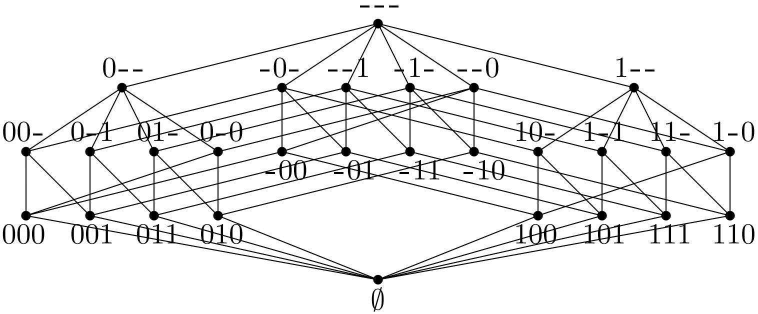


Example: Hypercubes

Theorem

The graph of $L(Q_n)$ admits a Hamiltonian cycle.

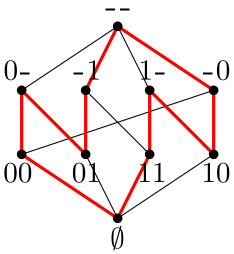


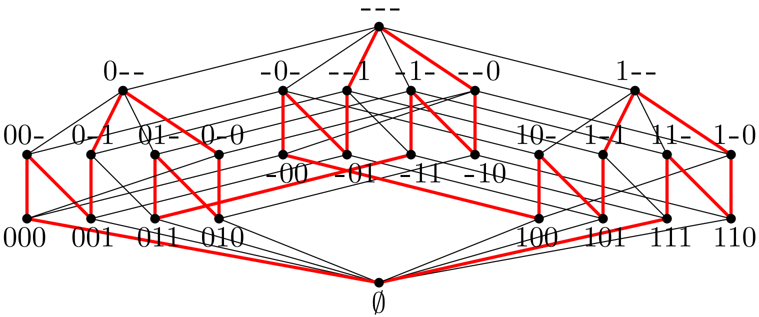


Example: Hypercubes

Theorem

The graph of $L(Q_n)$ admits a Hamiltonian cycle.

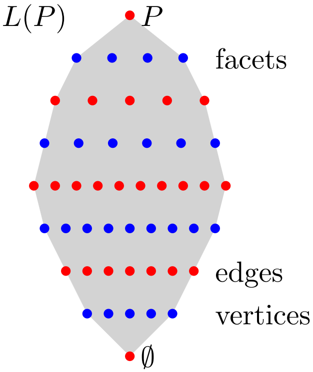




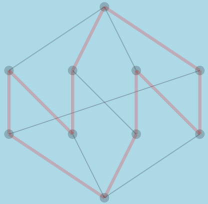
And in general...?

Conjecture

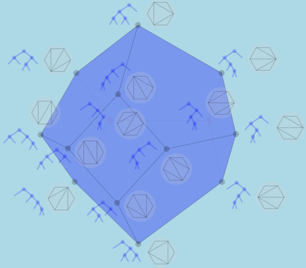
For any polytope P, the cover graph of $L(P)$ is Hamiltonian.



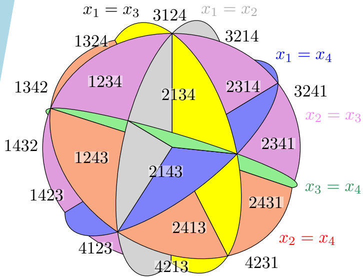
Introduction



Face Lattices



Regions in Hyperplane Arrangements



Zigzag framework

Zigzag framework

Aim: Gray codes for objects that are in bijection with certain subsets of permutations ("zigzag languages")

- binary reflected Gray code [Gray, 1953]
- Steinhaus-Johnson-Trotter algorithm [Steinhaus, Johnson, Trotter, 1960s]
- pattern avoiding permutations [Hartung, Hoang, Mütze, Williams, 2020]
- lattice quotients of the weak order (quotientopes) [Hoang, Mütze, 2021] [Pilaud, Santos, 2019]
- elimination trees of chordal graphs (graph associahedra of chordal graphs) [Cardinal, Merino, Mütze, 2022]
- (patter-avoiding) rectangulations (rectangulotopes) [Merino, Mütze, 2023] [Cardinal, Pilaud, 2025]
- ...

Regions of Hyperplane Arrangements

Regions of Hyperplane Arrangements

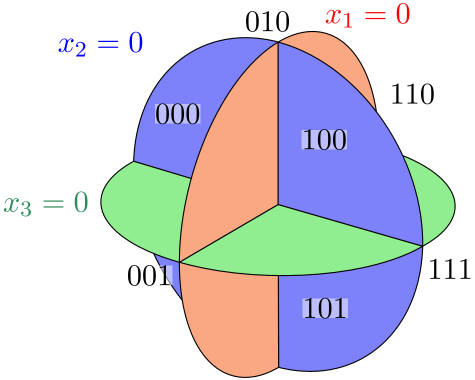
Hyperplane arrangements

A hyperplane arrangement $\mathcal{H}$ is a finite collection of hyperplanes (through the origin) in $\mathbb{R}^n$.

Regions of Hyperplane Arrangements

Hyperplane arrangements

A hyperplane arrangement $\mathcal{H}$ is a finite collection of hyperplanes (through the origin) in $\mathbb{R}^n$.

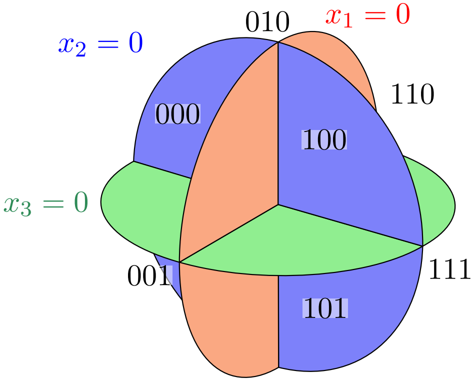


Regions of Hyperplane Arrangements

Hyperplane arrangements

A hyperplane arrangement $\mathcal{H}$ is a finite collection of hyperplanes (through the origin) in $\mathbb{R}^n$.

- **rank**: dimension of the space spanned by the normal vectors of hyperplanes in $\mathcal{H}$

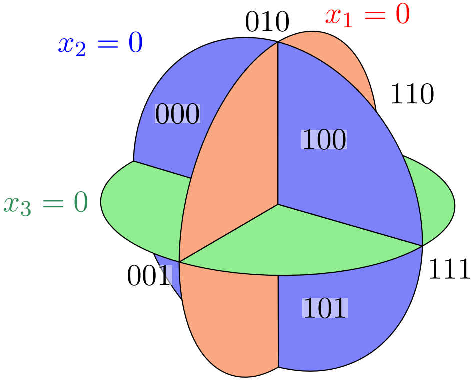


Regions of Hyperplane Arrangements

Hyperplane arrangements

A hyperplane arrangement $\mathcal{H}$ is a finite collection of hyperplanes (through the origin) in $\mathbb{R}^n$.

- **rank**: dimension of the space spanned by the normal vectors of hyperplanes in $\mathcal{H}$
- **regions**: connected components of $\mathbb{R}^n \setminus \mathcal{H}$

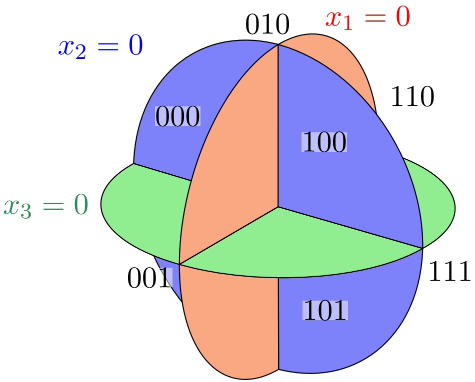


Regions of Hyperplane Arrangements

Hyperplane arrangements

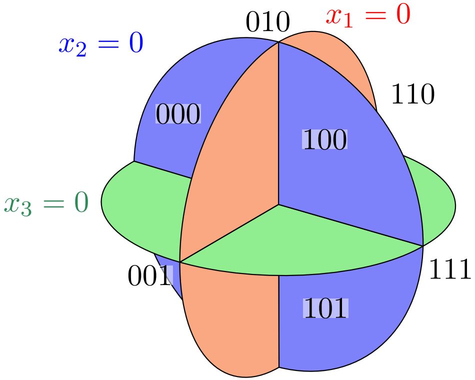
A hyperplane arrangement $\mathcal{H}$ is a finite collection of hyperplanes (through the origin) in $\mathbb{R}^n$.

- **rank**: dimension of the space spanned by the normal vectors of hyperplanes in $\mathcal{H}$
- **regions**: connected components of $\mathbb{R}^n \setminus \mathcal{H}$
- **graph of regions** $\mathcal{G}(\mathcal{H})$: vertices are regions, adjacent if separated by a single hyperplane

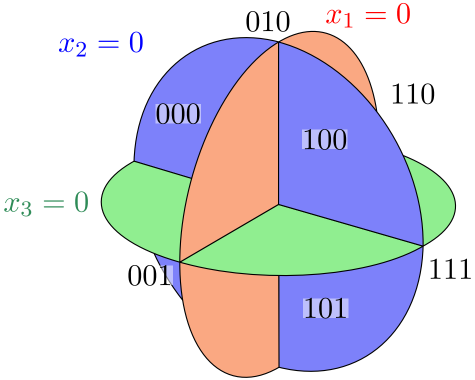


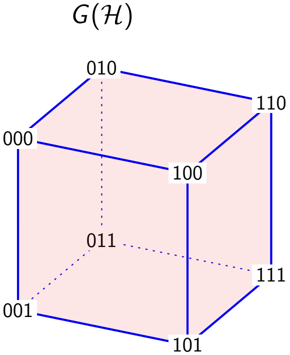
Example: Coordinate Arrangement

Example: Coordinate Arrangement



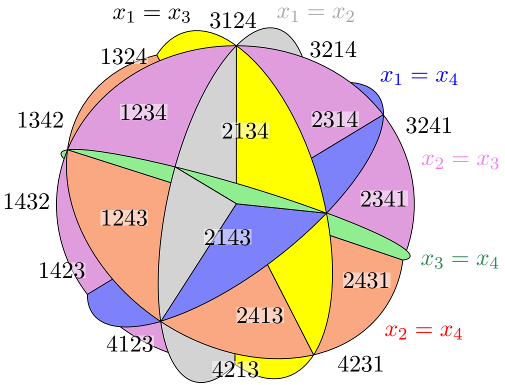
Example: Coordinate Arrangement



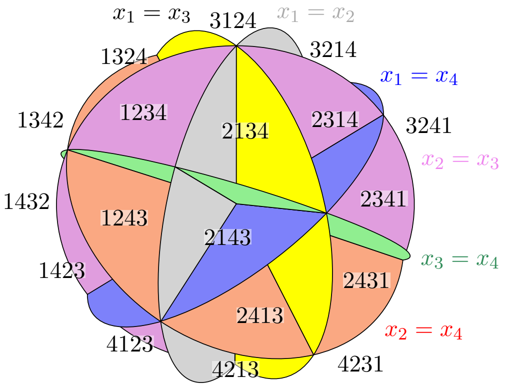


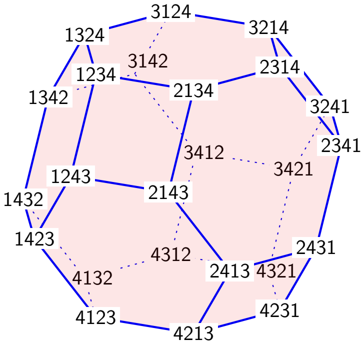
Example: Braid Arrangement

Example: Braid Arrangement



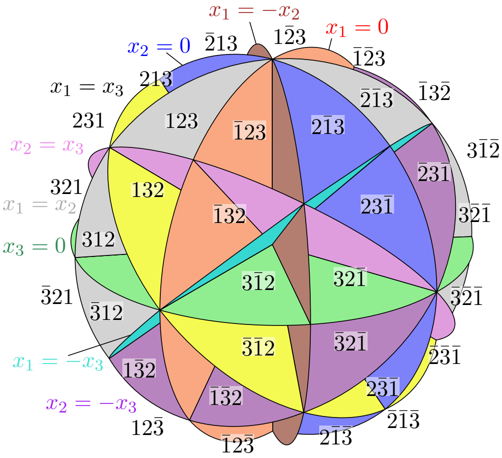
Example: Braid Arrangement



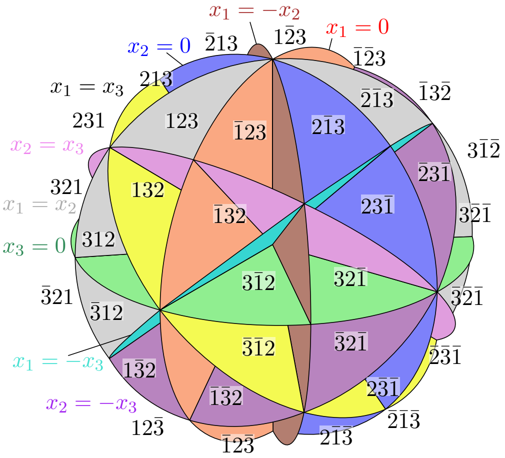


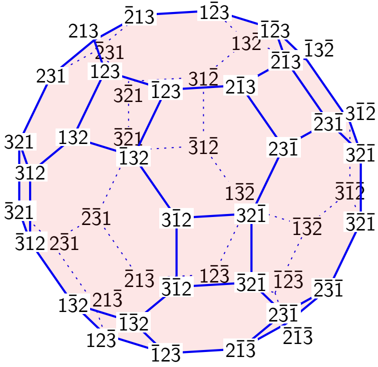
Example: Type B Arrangement

Example: Type B Arrangement



Example: Type B Arrangement



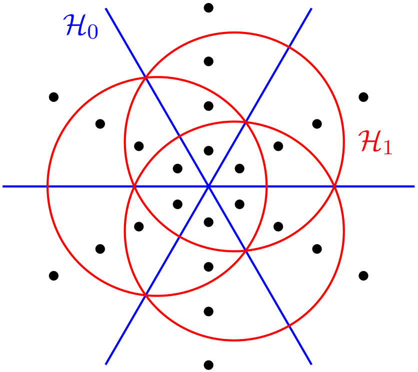


Supersolvable arrangements

Supersolvable arrangements

Supersolvable arrangements

A hyperplane arrangement $\mathcal{H}$ is supersolvable if

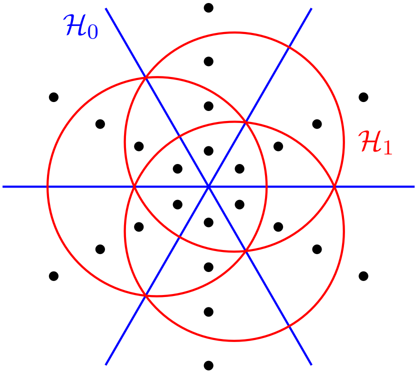


Supersolvable arrangements

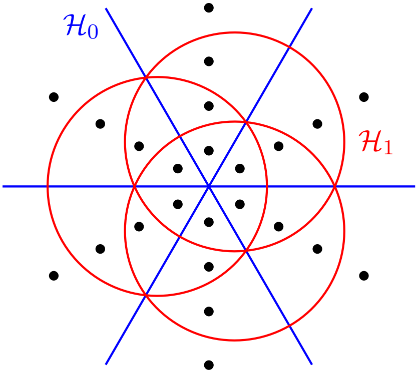
Supersolvable arrangements

A hyperplane arrangement $\mathcal{H}$ is supersolvable if

- it has rank ≤ 2, or



Supersolvable arrangements



Supersolvable arrangements

A hyperplane arrangement $\mathcal{H}$ is supersolvable if

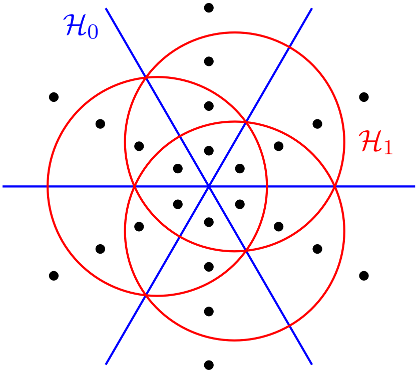
- it has rank ≤ 2, or
- $\mathcal{H} = \mathcal{H}_0 \,\dot\cup\, \mathcal{H}_1$, where

Supersolvable arrangements

Supersolvable arrangements

A hyperplane arrangement $\mathcal{H}$ is supersolvable if

- it has rank ≤ 2, or
- $\mathcal{H} = \mathcal{H}_0 \,\dot\cup\, \mathcal{H}_1$, where
  - $\mathcal{H}_0$ is a supersolvable arrangement of lower rank, and

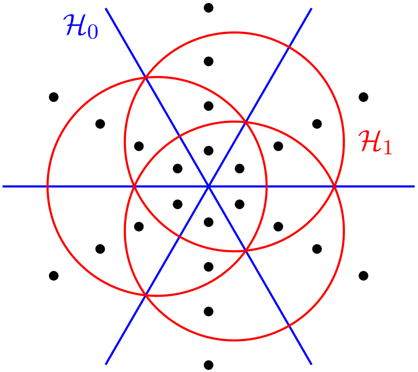


Supersolvable arrangements

Supersolvable arrangements

A hyperplane arrangement $\mathcal{H}$ is supersolvable if

- it has rank ≤ 2, or
- $\mathcal{H} = \mathcal{H}_0 \,\dot\cup\, \mathcal{H}_1$, where
  - $\mathcal{H}_0$ is a supersolvable arrangement of lower rank, and
  - for any $H, H' \in \mathcal{H}_1$, there is $H_0 \in \mathcal{H}_0$ such that $H \cap H' \subseteq H_0$.

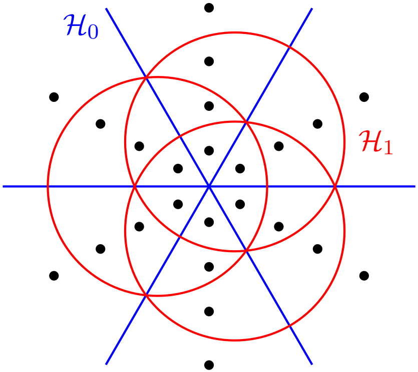


Hamiltonian cycles in supersolvable arrangements

Hamiltonian cycles in supersolvable arrangements

Theorem (B., Cardinal, McConville, Merino, Mütze, 2026)

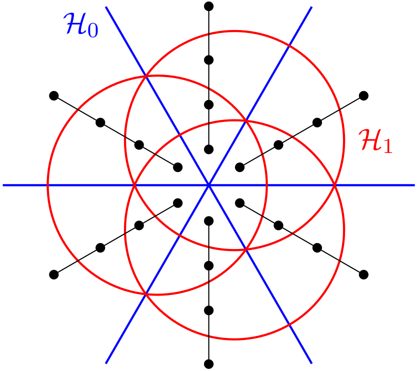
Let $\mathcal{H}$ be a supersolvable hyperplane arrangement of rank ≥ 2. Then $\mathcal{G}(\mathcal{H})$ has a Hamiltonian cycle.



Hamiltonian cycles in supersolvable arrangements

Theorem (B., Cardinal, McConville, Merino, Mütze, 2026)

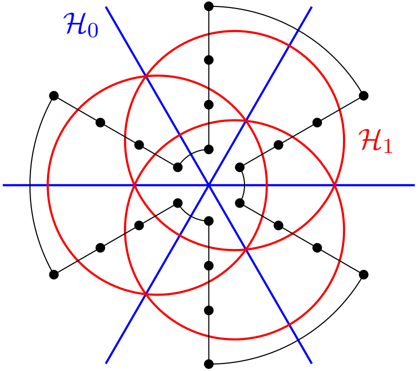
Let $\mathcal{H}$ be a supersolvable hyperplane arrangement of rank ≥ 2. Then $\mathcal{G}(\mathcal{H})$ has a Hamiltonian cycle.



Hamiltonian cycles in supersolvable arrangements

Theorem (B., Cardinal, McConville, Merino, Mütze, 2026)

Let $\mathcal{H}$ be a supersolvable hyperplane arrangement of rank ≥ 2. Then $\mathcal{G}(\mathcal{H})$ has a Hamiltonian cycle.



Results

Results

Coordinate arrangement:

Results

Coordinate arrangement:

- binary reflected Gray code

Results

Coordinate arrangement:

- binary reflected Gray code

Braid arrangement:

Results

Coordinate arrangement:

- binary reflected Gray code

Braid arrangement:

- Steinhaus-Johnson-Trotter algorithm

Results

Coordinate arrangement:

- binary reflected Gray code

Braid arrangement:

- Steinhaus-Johnson-Trotter algorithm
- lattice quotients of the weak order (quotients of the braid arrangement)

Results

Coordinate arrangement:

- binary reflected Gray code

Braid arrangement:

- Steinhaus-Johnson-Trotter algorithm
- lattice quotients of the weak order (quotients of the braid arrangement)
- acyclic orientations of chordal graphs (graphic arrangements)

Results

Coordinate arrangement:

- binary reflected Gray code

Braid arrangement:

- Steinhaus-Johnson-Trotter algorithm
- lattice quotients of the weak order (quotients of the braid arrangement)
- acyclic orientations of chordal graphs (graphic arrangements)

Type B arrangement:

Results

Coordinate arrangement:

- binary reflected Gray code

Braid arrangement:

- Steinhaus-Johnson-Trotter algorithm
- lattice quotients of the weak order (quotients of the braid arrangement)
- acyclic orientations of chordal graphs (graphic arrangements)

Type B arrangement:

- signed permutations

Results

Coordinate arrangement:

- binary reflected Gray code

Braid arrangement:

- Steinhaus-Johnson-Trotter algorithm
- lattice quotients of the weak order (quotients of the braid arrangement)
- acyclic orientations of chordal graphs (graphic arrangements)

Type B arrangement:

- signed permutations
- type B quotients (quotients of the type B arrangement)

Results

Coordinate arrangement:

- binary reflected Gray code

Braid arrangement:

- Steinhaus-Johnson-Trotter algorithm
- lattice quotients of the weak order (quotients of the braid arrangement)
- acyclic orientations of chordal graphs (graphic arrangements)

Type B arrangement:

- signed permutations
- type B quotients (quotients of the type B arrangement)
- symmetric triangulations (type B associahedron)

Results

Coordinate arrangement:

- binary reflected Gray code

Braid arrangement:

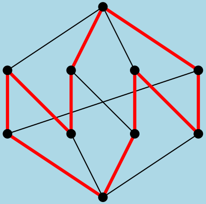
- Steinhaus-Johnson-Trotter algorithm
- lattice quotients of the weak order (quotients of the braid arrangement)
- acyclic orientations of chordal graphs (graphic arrangements)

Type B arrangement:

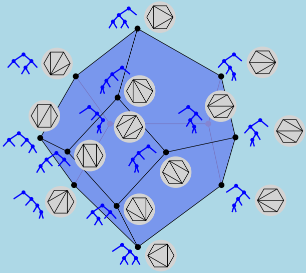
- signed permutations
- type B quotients (quotients of the type B arrangement)
- symmetric triangulations (type B associahedron)
- pattern avoiding signed permutations (zigzag framework)

Thank you!

Introduction



Face Lattices



Regions in Hyperplane Arrangements