Orbit-counting for groups acting on countable sets

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Overview

ORBIT-COUNTING

- We often want to count objects up to some symmetry, which can be formalized as orbit-counting.
- This is classical combinatorics for a group acting on a finite set.
- In the 1970s, Peter Cameron began considering group acting on a countable set.
- Given $G \curvearrowright X$, we naturally get an action of G on the n-subsets of X ($g.\{x_1, \ldots, x_n\} = \{g.x_1, \ldots, g.x_n\}$).
- Let $f_G(n)$ count the orbits of this action on n-subsets.
- We place the restriction that $f_G(n)$ is always finite.
- We will be particularly interested in the case when $f_G(n)$ is slow, and in jumps in the allowable behavior of $f_G(n)$.

EXAMPLES

- We will produce examples by taking $G = Aut(M)$ for a suitable countable structure M.
- If $M = (\mathbb{Q}, =)$ then $f_G(n) \equiv 1$. Similarly if $M = (\mathbb{Q}, <)$.
- If M is an equivalence relation with k infinite classes, then $f_G(n) \approx n^{k-1}$.
- If M is an equivalence relation with infinitely many infinite classes, then $f_G(n)$ is the partition function ($\approx e^{\sqrt{n}}$).
- If M is a has two refining equivalence relations with infinitely many infinite classes, then $f_G(n) \approx e^{n/\log n}$.
- If M is an equivalence relation with classes of size 2 and a linear order on the classes, then $f_G(n)$ is the Fibonacci sequence ($\approx 1.618^n$).

Examples (contd.)

- If M is the generic local order, then $f_G(n) \approx 2^n$.
- If M is a suitable tree-like structure, then $f_G(n)$ is the Catalan sequence ($\approx 4^n$).
- If M is a suitable permutation, then $f_G(n) = n!$ is the number of finite permutations of size n.
- If M is a suitable graph, then $f_G(n)$ is the number of graphs of size n ($\approx 2^{n^2}$).

COUNTING SUBSTRUCTURES

- This orbit-counting is equivalent to counting (up to isomorphism) n-substructures of nice countable structures.
- Given $G \curvearrowright X$, we may produce a relational structure M so that $Aut(M) \curvearrowright M$ has the same growth rate.
- The M produced has two nice properties indicating high symmetry.
  1. The finiteness condition on $f_{Aut(M)}(n)$ is equivalent to M being *ω-categorical*: it is uniquely determined by its first-order theory and by being countable.
  2. The M produced is *homogeneous*: orbits on n-subsets correspond to isomorphism types of n-substructures.
- We let $f_M(n)$ count the n-substructures of M.
- Our original problem is equivalent to: For ω-categorical and homogeneous M, understand $f_M(n)$.

The model-theoretic approach

- Model theory provides a series of “dividing lines” separating tame from wild behavior.
- Wild behavior is witnessed by the structure encoding a particular complicated configuration.
- Tame behavior ideally corresponds to a well-behaved independence notion, allowing a recursive decomposition of tame structures into simple independent parts.
- These dividing lines are often considered successively, accumulating more and more information.

Some early results on growth rate

- $f_M(n)$ is weakly increasing [Cam76], [Pou76]
- Classification of M such that $f_M \equiv 1$ [Cam76]
- If $f_M(n)$ is not bounded above by a polynomial, then it is bounded below by the partition function ($\approx e^{\sqrt{n}}$) [Mac85a]
- If M is primitive and $f_M(n) \not\equiv 1$, then $f_M(n) \geq (\sqrt[5]{2})^n / poly(n)$ [Mac85b]
- Under a further assumption, either $f_M(n)$ is slower than $2^{n^{1+\epsilon}}$ or is at least $2^{O(n^2)}$ [Mac87].

Conjectures and recent results

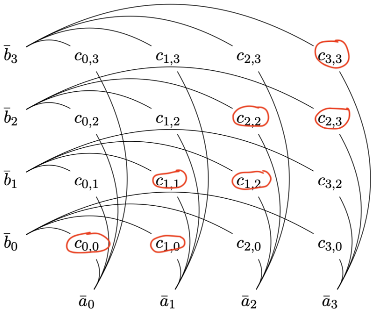
Conjecture (Cameron, Macpherson)

1. *If $f_M(n)$ is bounded above by a polynomial, then $f_M(n) \sim cn^k$ for some $c > 0, k \in \mathbb{N}$.*
2. *Suppose $f_M(n)$ is not bounded above by a polynomial, but is bounded above by $e^{n^{1-\epsilon}}$ for some $\epsilon > 0$. Then there are $k \in \mathbb{N}, \epsilon > 0$ such that*
$$e^{n^{1-1/k-\epsilon}} < f_M(n) < e^{n^{1-1/k+\epsilon}}$$
3. *Suppose M is primitive and $f_M(n) \not\equiv 1$. Then $f_M(n) \geq 2^n/poly(n)$.*

- Stronger form of (1) proved in [FT20] by algebraic combinatorics.
- For (3), Macpherson's base of $\sqrt[5]{2}$ was improved to ≈ 1.324 [Mer01] and then ≈ 1.576 [Sim18]

NIP

- A structure is *NIP* ("not the independence property") if it cannot encode arbitrary finite (bipartite) graphs.
- If M is not NIP, then $f_M(n) \geq 2^{O(n^2)}$.
- Let V be a countable-dimensional vector space over $\mathbb{F}_p$. There is a coloring of the points of V so the resulting structure is not NIP.
- $\{a_0, b_0, a_1, b_1, \ldots\}$ is linearly independent, and $c_{i,j} = a_i + b_j$.



Stability

- A structure is *stable* if it cannot encode an infinite linear order. This is stronger than NIP.
- Slow growth does not imply stability, e.g. $(\mathbb{Q}, <)$.
- But note the growth rate of $(\mathbb{Q}, <)$ is the same as $(\mathbb{Q}, =)$.

Theorem (Simon [Sim18])

Let M be an ω-categorical homogeneous structure such that $f_M(n) < \phi^n/poly(n)$.
Then there is a stable reduct M^* so that $f_M(n) = f_{M^*}(n)$.

Theorem (Simon [Sim18])

Let M be an ω-categorical homogeneous structure such that $\phi^n/poly(n) \leq f_M(n) < 2^n/poly(n)$. Then M is not primitive.

- So to prove the conjectures, it suffices to understand stable M.

Monadic stability

- Stability does not imply slow growth, e.g. vector spaces.
- M is *monadically stable* if every structure obtained by coloring of the points of M is stable.
- Vector spaces are stable, but we have seen they are not monadically stable (not even monadically NIP).

Proposition (B. [Bra22])

Let M be ω-categorical, homogeneous, and stable.

1. *If M is monadically stable, then $f_M(n)$ is slower than any exponential.*
2. *If M is not monadically stable, then $f_M(n)$ is faster than any exponential.*

- So it remains to understand monadically stable M.

The Baldwin-Shelah theorem

Theorem (Baldwin-Shelah [BS85])

The following are equivalent for a structure M.

1. *M is monadically stable.*
2. *M is stable and forking dependence is well-behaved (reduces to singletons and is transitive).*
3. *M is stable and does not code a grid on singletons.*
4. *M admits a tree-decomposition using countable substructures.*

- Lachlan [Lac92] classified ω-categorical monadically stable structures.

The main theorems

Theorem (B. [Bra22])

Let M be ω-categorical and homogeneous. If $f_M(n)$ is slower than $\phi^n/poly(n)$, then it is sub-exponential, and falls into one of the following cases.

1. *There are $c > 0, k \in \mathbb{N}$ so $f_M(n) \sim cn^k$.*
2. *There are $c > 0, k \in \mathbb{N}$ so $f_M(n) = \exp((c + o(1))(n^{1-1/k}))$.*
3. *There are $c > 0, k, r \in \mathbb{N}$ so $f_M(n) = \exp((c + o(1))(\frac{n}{\log^r(n)^{1/k}}))$.*

Furthermore, every such growth rate is realized by a monadically stable M.

Theorem (B. [Bra22])

Let M be ω-categorical, homogeneous, and primitive. If $f_M(n)$ is not constant 1, then $f_M(n)$ is at least $2^n/poly(n)$.

WHY MODEL THEORY?

- Model theory provides several dividing lines that can be used to explain jumps in complexity.
- Together with these dividing lines, there are various notions of independence to decompose tame structures.
- Model theory passes between different models of a theory, and allows asymptotic analysis on cardinals.

QUESTIONS

- What about when $f_M(n)$ is at most exponential?
- What about monadic NIP?
  - We conjecture $f_M(n)$ is at most exponential $\iff$ M is monadically NIP.
  - We prove an analogue of the Baldwin-Shelah theorem for monadic NIP [BL21].
- Why does *monadic* stability appear?
  - We show that in hereditary classes, stability/NIP collapses to monadic stability/NIP [BL22].
- How important is the group action?
  - Monadic stability/NIP specialize to important notions in structural graph theory in hereditary classes.

CHARACTERIZATIONS OF MONADIC NIP

Theorem (B.-Laskowski [BL21])

The following are equivalent for a complete theory T.

1. *T is monadically ~~stable~~ NIP.*
2. *~~T is stable and forking~~ Finite satisfiability dependence reduces to singletons and is transitive.*
3. *T ~~is stable and~~ does not code a grid on ~~singletons~~ tuples.*
4. *Models of T admit a ~~tree-~~decomposition into an ordered sequence of independent pieces.*
5. *Models of T have "linear rankwidth" bounded by some cardinal.*

- Like Baldwin-Shelah, this does not yield a good structure theory in the countable.

Conclusion

- (Monadic) stability/NIP provide broad generalizations of notions from finite combinatorics, capturing tree-like structure.
- The infinitary combinatorics and geometry of model-theoretic dividing lines is reflected in the finite.
- This can be used both to solve concrete problems and to provide a "geography" of classes of interest.

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