Twin-Width and Contraction Sequences - Set 1

Édouard Bonnet

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1 Twin-width of some graphs and graph families

A graph is a *cograph* if it does not contain the 4-vertex path as an induced subgraph.

Question 1. *Prove that cographs are exactly the graphs of twin-width 0.*

For $x, y \in \{\text{clique}, \text{stable}\}$, an x–y *half-graph* (of height n) has vertex set $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$, and edges between x_i and y_j whenever $i \leqslant j$. Additionally, $\{x_1, \ldots, x_n\}$ is a clique if $x = \text{clique}$ and $\{y_1, \ldots, y_n\}$ is a clique if $y = \text{clique}$.

Question 2. *What are the twin-widths of the clique–clique half-graph, clique–stable half-graph, and stable–stable half-graph of height n?*

A *unit interval graph* is the intersection graph of a collection of length-1 intervals of the real line. For the next question, it might be helpful to think about the twin-width of a *path of half-graphs* (and start by formalizing this notion).

Question 3. *What is the twin-width of unit interval graphs?*

Question 4. *Construct a simple explicit infinite family $G_1, G_2, \ldots$ such that the twin-width of G_i is at least i.*

Question 5. *Argue that there are infinite families of n-vertex graphs of twin-width $\Theta(n)$.*

The following question is open.

Open Question 1. *Are there n-vertex graphs of twin-width at least $\lceil n/2 \rceil$?*

We know of n-vertex graphs of twin-width $\frac{n-1}{2}$ for infinitely many n.

2 χ-boundedness

A graph class $\mathcal{C}$ is *χ-bounded* if there is a function f such that for all $G \in \mathcal{C}$, $\chi(G) \leqslant f(\omega(G))$ where $\chi(G)$ is the chromatic number of G, and $\omega(G)$, its clique number. A graph G is *triangle-free* if G does not have a 3-vertex clique, or equivalently if $\omega(G) \leqslant 2$.

Question 6. *Show that triangle-free graphs of twin-width at most d have chromatic number at most $d + 2$.*

Question 6 is difficult without any hint. (Ask one to your group leader or me!)

Question 7. *Show that classes of bounded twin-width are χ-bounded.*