Problem 1 (25 points)

- It is a sum of arithmetic sequence. Thus

\[ \sum_{k=1}^{n} k = \frac{1}{2}(n^n + 1) n^n. \]

- We will use the squeeze theorem. First, we observe following inequalities:

\[ (2n)^{2n} \leq \sum_{k=1}^{2n} k^{2n} \leq 2n(2n)^{2n}, \]

\[ \frac{1}{2} n^{2n} \leq \frac{1}{2}(n^n + 1) n^n \leq n^{2n}. \]

Thus

\[ \frac{1}{4} n^{\sqrt{\frac{1}{4} n}} = n^{\sqrt{\frac{1}{2} n^{2n}}} \leq a_n \leq n^{\sqrt{\frac{2n}{(2n)^2}}} = \frac{1}{4}. \]

It is well known that

\[ \lim_{n \to \infty} n^{\sqrt{\frac{1}{4} n}} = 1. \]

Now, using the squeeze theorem and previously showed inequalities we obtain that

\[ \lim_{n \to \infty} a_n = \frac{1}{4}. \]
Problem 2 (25 points)

(1) We are going to verify conditions of the implicit function theorem for equation

\[ F(x, y, z, w) = x^2 + 2y^2 + 3z^2 + 4w^2 - 10xyzw = 0 \]

and point \([1, 1, 1, 1]\).

\[ \text{– Since } F \text{ is polynomial we have } F \in C^1(\mathbb{R}^4), \]

\[ \text{– } F(1, 1, 1, 1) = 0, \]

\[ \text{– } F_y(1, 1, 1, 1) = -6 \neq 0, \text{ thus } f \text{ exists and belongs to } C^1. \]

(2)

\[ f_x(1, 1, 1) = -\frac{F_x(1, 1, 1, 1)}{F_y(1, 1, 1, 1)} = -\frac{4}{3}, \]

\[ f_z(1, 1, 1) = -\frac{F_z(1, 1, 1, 1)}{F_y(1, 1, 1, 1)} = -\frac{2}{3}, \]

\[ f_w(1, 1, 1) = -\frac{F_w(1, 1, 1, 1)}{F_y(1, 1, 1, 1)} = -\frac{1}{3}. \]

Using the chain rule we obtain

\[ T_z(1, 1, 1) = ((f_x + f_z + f_w)f_z)(1, 1, 1) = \frac{14}{9}. \]
**Problem 3** (25 points)

The domain of integration is compact and the integrand is continuous on it, hence the integral exists. Let’s use the substitution \( x = \varphi(y) \), where

\[
\varphi(y) = 2 \tan y, \quad \varphi'(y) = \frac{2}{\cos^2 y}.
\]

This substitution is valid, since \( \varphi \) is continuously differentiable and injective on \([\pi/4, \pi/3]\), and it maps \((\pi/4, \pi/3)\) onto \((2, 2\sqrt{2})\). Thus the integral equals

\[
\int_{\pi/4}^{\pi/3} \frac{1}{4 \sin^2 y \sqrt{4 + 4 \sin^2 y}} \cdot \frac{2}{\cos^2 y} \, dy = \frac{1}{4} \int_{\pi/4}^{\pi/3} \frac{\cos y}{\sin^2 y} \, dy.
\]

Now we use the substitution

\[
\psi(y) = \sin y, \quad \psi'(y) = \cos y,
\]

which is also valid, since \( \psi \) is continuously differentiable and injective on \([\pi/4, \pi/3]\). We thus obtain

\[
\frac{1}{4} \int_{\pi/4}^{\pi/3} \frac{1}{z^2} \, dz = \frac{1}{4} \left[ -\frac{1}{z} \right]_{\sqrt{2}}^{\sqrt{3}} = \frac{1}{2} \left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} \right).
\]
Problem 4 (25 points)

For both tasks it is useful to simplify the equation \( xv_1 + yv_2 + zv_3 = \mathbf{0} \). For fixed \( a, b \), this equation is equivalent to a system of linear equations, which can be simplified by elementary row operations as follows.

\[
\begin{pmatrix}
1 & 3 & b & 0 \\
1 & a + 3 & b - 2a & 0 \\
2 & 7 & 2b - 1 & 0 \\
1 & 5 & 2b - 4 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 3 & b & 0 \\
0 & a - 2a & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 2 & b - 4 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
1 & 3 & b & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & -a & 0 \\
0 & 0 & b - 2 & 0
\end{pmatrix}
\]

(i) The set \( \{v_1, v_2, v_3\} \) is linearly dependent if and only if the equation \( xv_1 + yv_2 + zv_3 = \mathbf{0} \) admits a nontrivial solution \((x, y, z) \neq (0, 0, 0)\). As is readily seen from the simplified form, this happens exactly when \( a = 0 \) and \( b = 2 \). Therefore, the only pair for which the given set is linearly dependent is \((a, b) = (0, 2)\).

(ii) If one of the vectors is a linear combination of the other two, then the set \( \{v_1, v_2, v_3\} \) is linearly dependent. Thus, necessarily, \((a, b) = (0, 2)\). In this case, one of the solutions to the system is \((x, y, z) = (-5, 1, 1)\), therefore \(-5v_1 + v_2 + v_3 = \mathbf{0}\). A simple algebra gives

\[
v_1 = \frac{1}{5}v_2 + \frac{1}{5}v_3, \quad v_2 = 5v_1 - v_3, \quad v_3 = 5v_1 - v_2.
\]