Problem 1 (25 points)

We are going to use the Fubini theorem. Therefore it is convenient for us to write

\[ M = M_1 \cup M_2 = \{0 < x \leq 1, 0 < y < x\} \cup \{1 < x < 2, 0 < y < 2 - x\} . \]

Now

\[
\int_{M_1} xy \, dx \, dy = \int_0^1 \left( \int_0^x xy \, dy \right) \, dx = \int_0^1 x \left[ \frac{y^2}{2} \right]_0^x \, dx \\
= \int_0^1 \frac{x^3}{2} \, dx = \left[ \frac{x^4}{8} \right]_0^1 = \frac{1}{8}
\]

and

\[
\int_{M_2} xy \, dx \, dy = \int_1^2 \left( \int_0^{2-x} xy \, dy \right) \, dx = \int_1^2 x \left[ \frac{y^2}{2} \right]_0^{2-x} \, dx \\
= \int_1^2 \frac{x(2-x)^2}{2} \, dx = \int_1^2 \frac{x^3}{2} - 2x^2 + 2x \, dx = \left[ \frac{x^4}{8} - \frac{2x^3}{3} + x^2 \right]_1^2 \\
= 2 - \frac{16}{3} + 4 - \frac{1}{8} + \frac{2}{3} - 1 = \frac{5}{24} .
\]

Thus

\[
\int_M xy \, dx \, dy = \int_{M_1} xy \, dx \, dy + \int_{M_2} xy \, dx \, dy = \frac{1}{8} + \frac{5}{24} = \frac{1}{3} .
\]
Problem 2 (25 points)

(i) The domain of definition of the function $f$ is

$$D(f) = (-\infty, 1) \cup (1, 2) \cup (2, \infty).$$

(ii) Since a ratio of two continuous functions is a continuous function on a set where it is defined, and $\ln$ is continuous, we have that $f$ is continuous in $D(f)$.

(iii) $\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} = 0$, $\lim_{x \to 1} = -\infty$ a $\lim_{x \to 2} = \infty$.

(iv) The first derivative of $f$ in $x \in D(f)$ is

$$f'(x) = -\frac{1}{(x-1)(x-2)}.$$

Thus

- $f'(x) < 0$, i.e. $f$ is strictly decreasing, in the interval $(-\infty, 1)$.
- $f'(x) > 0$, i.e. $f$ is strictly increasing, in the interval $(1, 2)$.
- $f'(x) < 0$, i.e. $f$ is strictly decreasing, in the interval $(2, \infty)$.

Since $f'$ exists and $f' \neq 0$ in $D(f)$, the function $f$ has no points of local minima and local maxima in $D(f)$. As $f$ is not bounded in $D(f)$ from below and from above, it does not attain in $D(f)$ its global maximum and global minimum.

(v) The second derivative in $x \in D(f)$ is

$$f''(x) = \frac{2x-3}{(x^2-3x+2)^2}.$$

Thus

- $f''(x) < 0$, i.e. $f$ is concave, in the interval $(-\infty, 1)$.
- $f''(x) < 0$, i.e. $f$ is concave, in the interval $(1, \frac{3}{2})$.
- $f''(x) > 0$, i.e. $f$ is convex, in the interval $(\frac{3}{2}, 2)$.
- $f''(x) > 0$, i.e. $f$ is convex, in the interval $(2, \infty)$.

(vi) The function $f$ has the asymptote $v(x) = 0$ both in $-\infty$ and $\infty$.

(vii) The corresponding plot is
Problem 3 (25 points)

Put $g(x, y, z) = xyz$. Since exponential is increasing, it is sufficient to find maximum and minimum of $g$ on $M$. Denote $\Phi(x, y, z) = x^2 + 2y^2 + 3z^2 - 30$. Since $\Phi$ is continuous and $M = \Phi^{-1}(\{0\})$ we have that $M$ is closed. By, $\Phi(x, y, z) = 0$ we obtain that $x^2 \leq 30, y^2 \leq 30, z^2 \leq 30$. Thus, $M$ is bounded. So, $M$ is compact. Using the continuity of $g$ we have that $g$ attains its maximum and minimum on $M$. We use Lagrange Multiplier Theorem on set $M$ and function $g$. We can use this Theorem since $g, \Phi \in C^1(\mathbb{R}^3)$ and $\nabla \Phi(x, y, z) = (2x, 4y, 6z) = (0, 0, 0)$ if and only if $(x, y, z) = (0, 0, 0) \notin M$. Now, we find $x, y, z \in \mathbb{R}$ such that there exists $\lambda \in \mathbb{R}$ such that $\nabla (g - \lambda \Phi)(x, y, z) = (0, 0, 0)$, $\Phi(x, y, z) = 0$. The solutions are $(x, y, z) = (0, 0, \pm \sqrt{10}), (0, \pm \sqrt{15}, 0), (\pm \sqrt{30}, 0, 0), (\pm \sqrt{10}, \pm \sqrt{5}, \pm \sqrt{5 \over 3})$. Now, we see that the maximum of $f$ on $M$ is $\exp(10 \sqrt{5 \over 3})$ and minimum is $\exp(-10 \sqrt{5 \over 3})$. 
Problem 4 (25 points)

By using basic properties of determinants and Laplace’s formula we get

\[
\begin{vmatrix}
  a + 2 & 1 & 0 & 2 \\
  2 & 2 & 2 & 5 \\
  a + 3 & b & 0 & 3 \\
  1 & 2 & 2 & 4 \\
\end{vmatrix}
= 
\begin{vmatrix}
  a + 2 & 1 & 0 & 2 \\
  1 & 0 & 0 & 1 \\
  a + 3 & b & 0 & 3 \\
  1 & 2 & 2 & 4 \\
\end{vmatrix}
= -2 \begin{vmatrix}
  a + 2 & 1 & 2 \\
  1 & 0 & 1 \\
  a + 3 & b & 3 \\
  a & b & 3 \\
\end{vmatrix}

= 2 \begin{vmatrix}
  a & 1 \\
  a & b \\
\end{vmatrix}
= 2(ab - a) = 2a(b - 1)
.

The determinant of \( A \) is equal to \( 2a(b - 1) \). Matrix \( A \) is invertible if and only if its determinant is non-zero, that is, if and only if \( a \neq 0 \) and \( b \neq 1 \).