**Assignment A — Solutions**

**Problem 1** (25 points)

First, determine \( M = \{ [x, y] : x \in (0, 1), y \in (x^2, x) \} \). It is easy to see that the integral exists. We will use Fubini’s theorem to calculate it.

\[
\int_M (4y^3 - xy + 4x^3) \, dx \, dy = \int_0^1 \left[ \int_{x^2}^x (4y^3 - xy + 4x^3) \, dy \right] \, dx.
\]

Now calculate the inner integral

\[
\int_{x^2}^x (4y^3 - xy + 4x^3) \, dy = -x^8 - \frac{7}{2} x^5 + 5x^4 - \frac{1}{2} x^3.
\]

Integrating this function on the interval (0, 1) leads to the result. We have

\[
\int_M (4y^3 - xy + 4x^3) \, dx \, dy = \int_0^1 \left( -x^8 - \frac{7}{2} x^5 + 5x^4 - \frac{1}{2} x^3 \right) \, dx = \frac{13}{72}.
\]

**Problem 2** (25 points)

The function \( f \) is given by

\[
f(x) = \sqrt{x^2 - \frac{1}{x}}
\]

(i) The domain \( D(f) \) of \( f \) is the set of all \( x \) such that the expression \( x^2 - \frac{1}{x} \) is non-negative. After a simple calculation we get

\[
D(f) = (-\infty, 0) \cup (1, \infty).
\]

(ii) From the theorem on the continuity of the sum of two continuous functions and from the continuity of \( \sqrt{y} \) it follows that \( f \) is continuous at every point of the domain \( D(f) \).

(iii) We have \( \lim_{x \to -\infty} f(x) = \lim_{x \to 0^-} f(x) = \lim_{x \to \infty} f(x) = \infty \) and \( f(1) = 0 \).

(iv) A simple calculation yields

\[
f'(x) = \frac{1}{2} \left( x^2 - \frac{1}{x} \right)^{-1/2} \left( 2x + \frac{1}{x^2} \right)
\]

on \((-\infty, 0) \cup (1, \infty)\).

The sign of the derivative changes as follows:

- \( f'(x) < 0 \), and hence \( f \) is decreasing on the interval \((-\infty, -1/\sqrt{2})\);
- \( f'(x) > 0 \), and hence \( f \) is increasing on the interval \((-1/\sqrt{2}, 0)\);
- \( f'(x) > 0 \), and hence \( f \) is increasing on the interval \((1, \infty)\).
Because \( f \) is right-continuous at 1 and because \( \lim_{x \to 1^+} f'(x) = \infty \), \( f'(1+) = \infty \): this fact can be used in sketching the graph.

The function \( f \) has a local minimum at the point \(-1/\sqrt{2}\); the functional value at this point is \( \sqrt{3}/\sqrt{2} \). The function \( f \) is not bounded from above on \( D(f) \), therefore it does not attain its maximum. Its minimum is attained at 1 and \( f(1) = 0 \).

(v) The function \( f \) has an asymptote at \( \infty \) if and only if there exist finite limits \( \lim_{x \to \infty} f(x) = a \) and \( \lim_{x \to \infty} (f(x) - ax) = b \). The asymptote is given by the function \( ax + b \). The case of \( -\infty \) is analogous.

It is easy to find out that there is an asymptote at \( -\infty \) of the form \( v(x) = -x \). There is also an asymptote at \( \infty \) of the form \( w(x) = x \).

(vi) A sketch of the graph based on the previous calculations:

\[
\begin{align*}
\text{Problem 3} & \quad (25 \text{ points}) \\
\text{Because the function } f \text{ is continuous on } \mathbb{R}^3 \text{ (it even belongs to the class } C^\infty(\mathbb{R}^3)) \text{ we can use the identities } \sup_M f = \max_M f \text{ and } \inf_M f = \min_M f. \text{ If the function } f \text{ attains its supremum at a point belonging to } M, \text{ } f \text{ attains its maximum on } M \text{ at this point, too.} \\
\text{We will investigate the function } f \text{ on the set } \bar{M}. \text{ The set is bounded and closed in } \mathbb{R}^3 \text{ and } f \text{ is continuous. Hence } f \text{ attains its maximum an minimum on } M. \text{ Suspicious points are points in } M \text{ identified by the Lagrange multiplier method, and points in } \bar{M} \text{ with at least one zero coordinate. Consider the sets } M_i, i = 1, 2, 3: \\
M_1 = \{(y, 1 - y); y \in \langle 0, 1 \rangle\}, \quad M_2 = \{[1 - y, y, 0]; y \in \langle 0, 1 \rangle\} \\
\text{and } M_3 = \{[x, 0, 1 - x]; x \in \langle 0, 1 \rangle\}. \\
\text{On } M_3, \text{ } f \equiv 0. \text{ Because } f \text{ is non-negative on } \bar{M}, \text{ all points in } M_3 \text{ are points of minima of } f \text{ on } \bar{M}. \text{ On the sets } M_1 \text{ and } M_2, \text{ the function } f \text{ is equal to } y^2(1 - y)^2. \text{ This function of a single argument has “suspicous” points } 0, 1/2 \text{ a 1 in the interval } \langle 0, 1 \rangle. \text{ Only the point } 1/2 \text{ is relevant. There are}
\end{align*}
\]
two suspicious points on the sets \( M_1 \) and \( M_2: [0, 1/2, 1/2] \) and \([1/2, 1/2, 0]\). The function \( f \) attains the value 1/16 at these points.

Now investigate suspicious points in \( M \) by the Lagrange multiplier method. First, \( \nabla (x + y + z - 1) = [1, 1, 1] \), this point is not in \( M \). The Lagrange function has the form \( L(x, y, z, \lambda) = x^2y^2 + y^2z^2 + \lambda(x + y + z - 1) \). To find the value of the multiplier and coordinates of suspicious points we formulate a system of equations

\[
2xy^2 + \lambda = 0, \quad 2y(x^2 + z^2) + \lambda = 0, \quad 2yz^2 + \lambda = 0, \quad x + y + z = 1.
\]

The system has a single solution in \( M \), namely \([1/4, 1/2, 1/4]\). Its functional value is 1/32.

To summarize: \( f \) attains its maximum 1/16 in \( \bar{M} \) at \([0, 1/2, 1/2]\) a \([1/2, 1/2, 0]\) and its minimum 0 at \([x, 0, 1-x]\), \( x \in (0, 1) \). Because none of these points is in \( M \), \( f \) does not attain its maximum or minimum on \( M \). However, \( \sup_M f = 1/16, \inf_M f = 0 \).

**Problem 4** (25 points)

By a sequence of suitable transformations we obtain matrices

\[
\begin{pmatrix}
1 & 0 & 5 & -1 \\
0 & 3 & -2 & 5 \\
0 & 9 & -6 & a + 2 \\
0 & 15 & b - 5 & 25
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 5 & -1 \\
0 & 3 & -2 & 5 \\
0 & 0 & 0 & a - 13 \\
0 & 0 & b + 5 & 0
\end{pmatrix}, \quad \begin{pmatrix}
1 & 0 & 5 & -1 \\
0 & 3 & -2 & 5 \\
0 & 0 & 0 & a - 13 \\
0 & 0 & 0 & a - 13
\end{pmatrix}.
\]

It is now obvious that \( r(A) = 2 \) if \( a = 13 \) and \( b = -5 \), \( r(A) = 3 \) if \( a \neq 13 \) and \( b = -5 \) or \( a = 13 \) and \( b \neq -5 \), and \( r(A) = 4 \) if \( a \neq 13 \) and \( b \neq -5 \).