Assignment A — Solutions

Problem 1 (25 points)

The integral of \( f(x, y) = 1 \) over the set \( M (= \text{the area of } M) \) exists. Calculation of the area can be done by means of Fubini’s theorem.

1st approach

Calculate the \( x \) coordinates of the points \( A_1 \) and \( A_2 \), in which the two curves intersect. We obtain \( x_1 = x_2 = 1/2 \). The projection \( M_x \) of the set \( M \) onto the \( x \)-axis is \((-1/2, 1)\). From these results we have

\[
\int_M 1 \, dx \, dy = \int_{-1/2}^{1/2} \left( \int_{-\sqrt{2x+1}}^{\sqrt{2x+1}} \, dy \right) \, dx + \int_{1/2}^{1} \left( \int_{-2\sqrt{1-x}}^{2\sqrt{1-x}} \, dy \right) \, dx = 2\sqrt{2}.
\]

2nd approach

Calculate the \( y \) coordinates of the points \( A_1 \) and \( A_2 \), in which the two curves intersect. We obtain \( y_{1,2} = \pm \sqrt{2} \). The projection \( M_y \) of the set \( M \) onto the \( y \)-axis is \((-\sqrt{2}, \sqrt{2})\). From these results we have

\[
\int_M 1 \, dx \, dy = \int_{-\sqrt{2}}^{\sqrt{2}} \left( \int_{y^2-1}^{-y^2+4} \, dx \right) \, dy = 2\sqrt{2}.
\]
Problem 2 (25 points)

The function $f$ is continuous on $\mathbb{R}^2$, so it is continuous on $M$. The set $M$ is bounded and closed in $\mathbb{R}^2$. According to the theorem claiming that a continuous function on a compact set has a maximum and minimum, $f$ attains on $M$ its maximum and minimum. The points “suspicious” of being the points of extrema are boundary points of $M$ and critical points of $f$ located in the interior of $M$ ($M^0$).

**Critical points in the interior of $M$:**

Solving the system of the equations

$$\frac{\partial f}{\partial x}(x, y) = 2x - y = 0, \quad \frac{\partial f}{\partial y}(x, y) = 2y - x = 0$$

we find one critical point in $M^0$, namely $[0, 0]$. We have $f(0, 0) = 0$.

**Examination of $f$ on the boundary of $M$:**

At the corner points of $M$ (i.e., $[1, 0]$, $[-1, 0]$, $[0, 1]$, and $[0, -1]$) the value of $f$ equals 1.

It remains to study the problem on the sets $H_1 = \{[x, y]; x \in (0, 1), y = 1 - x\}$, $H_2 = \{[x, y]; x \in (0, 1), y = 1 - x\}$, $H_3 = \{[x, y]; x \in (-1, 0), y = 1 + x\}$ and $H_4 = \{[x, y]; x \in (-1, 0), y = -1 - x\}$. It is easy to see that there are four other suspicious points: $[1/2, 1/2]$, $[-1/2, -1/2]$, $[-1/2, 1/2]$, and $[1/2, -1/2]$. The values of $f$ are $1/4$ at the first two points and $3/4$ at the last two points.

To summarize, the function $f$ attains its minimum on $M$ at the point $[0, 0]$ with $\min_M f = f(0, 0) = 0$ and the function $f$ attains its maximum on $M$ at the points $[1, 0]$, $[-1, 0]$, $[0, 1]$, and $[0, -1]$, with $\max_M f = 1$. 
Problem 3 (25 points)

(i)

Likelihood function:

\[ L(\theta) = \theta^{-n} \left( \prod_{i=1}^{n} X_i \right) \exp \left\{ -\frac{1}{2\theta} \sum_{i=1}^{n} X_i^2 \right\} \]

Log-likelihood:

\[ \ell(\theta) = \sum_{i=1}^{n} \log X_i - n \log \theta - \frac{1}{2\theta} \sum_{i=1}^{n} X_i^2 \]

Score statistic:

\[ U(\theta) = \frac{\partial \ell(\theta)}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{2\theta^2} \sum_{i=1}^{n} X_i^2 \]

Score function:

\[ U_i(\theta) = -\frac{1}{\theta} + \frac{X_i^2}{2\theta^2} \]

Likelihood equation:

\[ U(\hat{\theta}_n) = 0 \quad \text{and thence} \quad \hat{\theta}_n = \frac{1}{2n} \sum_{i=1}^{n} X_i^2 \]

The unique solution of the likelihood equation is \( \hat{\theta}_n = \frac{1}{2n} \sum_{i=1}^{n} X_i^2 \). It is the maximum likelihood estimator because \( \ell(\theta) \) is continuous and unbounded from below.

(ii)

We calculate the density of the random variable \( Y = X^2 \), where \( X \) has the density \( f(x; \theta) \). Since \( P[X < 0] = 0 \), the transformation is one-to-one. The inverse transformation is \( x = \sqrt{y} \), the Jacobian is \( \frac{1}{2\sqrt{y}} \). The random variable \( Y \) has the density \( g(y; \theta) = (2\theta)^{-1} \exp\{y/(2\theta)\} \). Its distribution is \( \text{Exp}(1/(2\theta)) \), expectation \( EY = EX_i^2 = 2\theta \), variance \( \text{var} Y = \text{var} X_i^2 = 4\theta^2 \).

Because \( X_i \) are independent and \( 2n\hat{\theta}_n = \sum_{i=1}^{n} X_i^2 \), we have immediately

\[ 2n\hat{\theta}_n \sim \Gamma \left( \frac{1}{2\theta}, n \right), \quad E \hat{\theta}_n = \frac{1}{2n} n EX_i^2 = \theta, \quad \text{and} \quad \text{var} \hat{\theta}_n = \frac{1}{4n^2} n \text{var} X_i^2 = \frac{\theta^2}{n}. \]

(iii)

Since \( \hat{\theta}_n \) is the maximum likelihood estimator and regularity conditions are satisfied, its asymptotic distribution is \( \sqrt{n}(\hat{\theta}_n - \theta) \overset{D}{\to} \text{N}(0, 1/I(\theta)) \), where the Fisher information is \( I(\theta) = E \frac{\partial U_i(\theta)}{\partial \theta} = E X_i^2/\theta^3 - 1/\theta^2 = 1/\theta^2 \). We have

\[ \sqrt{n}(\hat{\theta}_n - \theta) \overset{D}{\to} \text{N}(0, \theta^2). \]
Problem 4 (25 points)

By *portfolio* we mean a group of financial assets. It is represented by fractions (allocation, diversification) of the investor’s unit wealth invested in the individual assets. Formally, if we denote \( \mathbf{x} = (x_1, \ldots, x_N)^T \) these fractions, then with the invested wealth of amount 1, the relation \( x_1 + \cdots + x_N = 1 \) must hold.

(i) Let us denote the weights in portfolio \( x_1 \) and \( x_2 \). Obviously, \( x_1 + x_2 = 1 \) must hold. It follows for the expected return of portfolio

\[
\mathbf{r}_P = 10x_1 + 8x_2 = 10x_1 + 8(1 - x_1) = 2x_1 + 8 = 9,
\]

and thus \( x_1 = x_2 = 1/2 \).

(ii) Let us denote the weights in portfolio \( x_0, x_1, \) and \( x_2 \). Again, \( x_0 + x_1 + x_2 = 1 \) must hold. The expected return of portfolio is therefore

\[
\mathbf{r}_P = 6x_0 + 10x_1 + 8x_2 = 6 + 4x_1 + 2x_2.
\]

The desired return of portfolio is \( \mathbf{r}_P = 9 \), so that from last equation we get

\[
x_2 = \frac{1}{3}(3 - 4x_1).
\]

If the returns of the risky assets are \( R_1 \) a \( R_2 \), we get (after the substitution for \( x_2 \)) the variance of the return of the portfolio:

\[
\text{var}(x_1R_1 + x_2R_2) = x_1^2 \text{var}(R_1) + 2x_1x_2 \text{cov}(R_1, R_2) + x_2^2 \text{var}(R_2) = 4x_1^2 + 2x_1x_2 + 2x_2^2 = \frac{9}{2} - 9x_1 + 8x_1^2.
\]

This is a convex function so that its minimum is obtained by equating its derivative to zero. Since the derivative of the last expression is \( 16x_1 - 9 \), we get \( x_1 = 9/16 \). Back substitutions to the previous equations gives \( x_2 = 3/8, \) \( x_0 = 1/16 \).