Entrance examination, MFF UK in Prague

Study programme for Bachelor of Computer Science

2013, version A

For each of the following ten problems five possible answers a, b, c, d, e are offered. For each problem your task is to mark for each answer whether it is true or false, or whether the statement holds or does not hold. The duration of the test is **75 minutes**.

Awarding points. 10 points are assigned to each problem. You gain 10 points for a given problem if and only if you mark each of the five offered answers correctly.[1] For a given problem, if one of the answers is marked incorrectly you will be awarded 0 points, even if some of the other answers for the problem have been marked correctly. For each problem where no answer is marked incorrectly you obtain 2 points for each correctly marked answer. If you mark all five answers correctly, you gain the maximum score of 10 points.

How to mark answers and how to make a correction. The answer you choose should be marked by filling the corresponding circle. If you have already marked an answer and wish to make a correction, you can cancel your choice by making a large cross over the filled circle, and then correct it by filling the other circle. It is not possible to choose an answer again where the circle has been already crossed out. Answers marked in any other way will be regarded as non-marked. Notice in the following example that the answers in the last two columns are the same, as they differ only by the corrections made to them.

Example. As an example we show the scoring for four markings for the problem "The sum of 1 + 1 is":

		Answers		Answers		Answers		Answers	
		Yes	No	Yes	No	Yes	No	Yes	No
(a)	2	●	○	●	○	○	○	○	○
(b)	3	○	●	○	○	○	●	~~●~~	●
(c)	Less than 12	●	○	●	○	○	○	~~●~~	○
(d)	Positive number	●	○	○	○	●	○	●	~~●~~
(e)	1	○	●	●	○	○	●	~~●~~	●
Points:		**10**		**0**		**6**		**6**	

[1]A correctly marked answer is one where the right answer is Yes and you only mark Yes or the right answer is No and you only mark No. An incorrect answer is one where the right answer is Yes and you only mark No or the right answer is No and you only mark Yes. All other possibilities are regarded as being unanswered.

For each of the following problems decide which assertions hold and which do not (Yes = Holds, No = Does not hold). Notation for intervals: $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ and $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

1. Find the set M of all solutions of the inequality $\frac{x-2}{x-3} \leq 0$ in the real domain.

a) The set M is a closed interval.
b) $M = (-\infty, 2]$
c) $M \cap (2, 4) \neq \emptyset$
d) $M \subset (0, \infty)$
e) $M \cap (-1, 1) \neq \emptyset$

2. Find the set M of all solutions of the inequality $x^2 - 5 - |x + 1| > 0$ in the real domain.

a) All solutions are strictly positive.
b) $(4, \infty) \subset M$
c) All solutions are strictly less than $\sqrt{17}$.
d) The number $\frac{1}{2}(\sqrt{17} - 1)$ is a solution of the inequality.
e) $M \cap (-1, 2) \neq \emptyset$

3. Compute the distance d of the point $(2, 3)$ from the line passing through the points $(-3, 3)$ and $(-7, 0)$.

a) $d > \frac{5}{2}$
b) $d > 4$
c) $d < \frac{8}{3}$
d) $d < \frac{10}{3}$
e) $d \in [\frac{1}{2}, \sqrt{10}]$

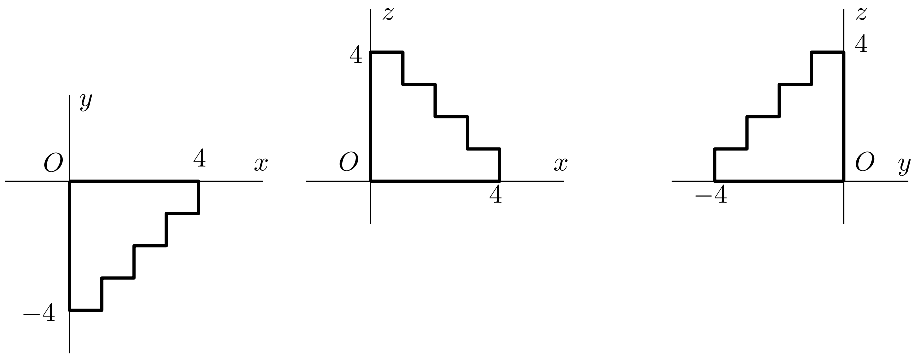
4. The numbers 1339, 1080, and 1741 share several properties: each number is strictly positive, it is an integer, it has four digits, it starts with the digit 1 and contains exactly two identical digits. Denote by P the size of the set of all numbers with these properties.

a) $P \in [121, 358]$
b) $P \in [221, 458]$
c) $P \in [321, 499]$
d) $P \in [430, 510]$
e) $P > 510$

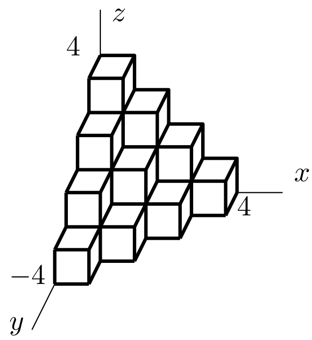
5. Solve the equation $3 \sin^2 x + \cos^2 x = 2\sqrt{2} \sin x$ in the real domain. Denote the set of all solutions by M.

a) The set M has exactly one element.
b) $\frac{401\pi}{4} \in M$
c) $M \cap (-\frac{\pi}{6}, \frac{\pi}{6}) = \emptyset$
d) $M \cap (0, \frac{\pi}{3}) = \emptyset$
e) The set M is empty.

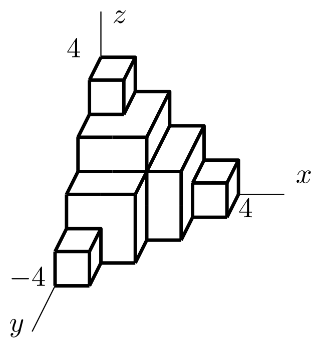
6. The body T has the following projections onto the three planes parallel to a pair of axes. The point O denotes the origin of the coordinate system. Decide which assertions hold.



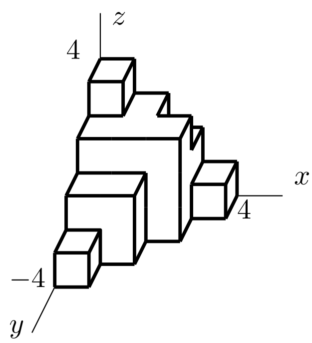
a) The body T is necessarily of the form shown in the following picture:



b) The body T can have the form shown in the following picture:



c) The body T can have the form shown in the following picture:



d) The volume of the body can be 20.
e) The volume of the body is at most 23.

7. For real numbers a, b, c consider the following relations:

$$|a+b+c| = |a|+|b|+|c|, \qquad \text{(X)}$$
$$ab+ac+bc \geq 0. \qquad \text{(Y)}$$

a) If real numbers a, b, c satisfy (X), then they necessarily satisfy (Y).
b) If real numbers a, b, c satisfy (Y), then they necessarily satisfy (X).
c) The condition (X) is satisfied for all real numbers a, b, c.
d) For all real numbers a, b, c the following holds: if the largest number among a, b, c is strictly negative, then (Y) is satisfied.
e) There exist real numbers a, b, c such that the condition (X) is not satisfied.

8. Solve the following system of equations with real parameter λ in the real domain:

$$x + \lambda y = 1,$$
$$\lambda x + 2y = \lambda.$$

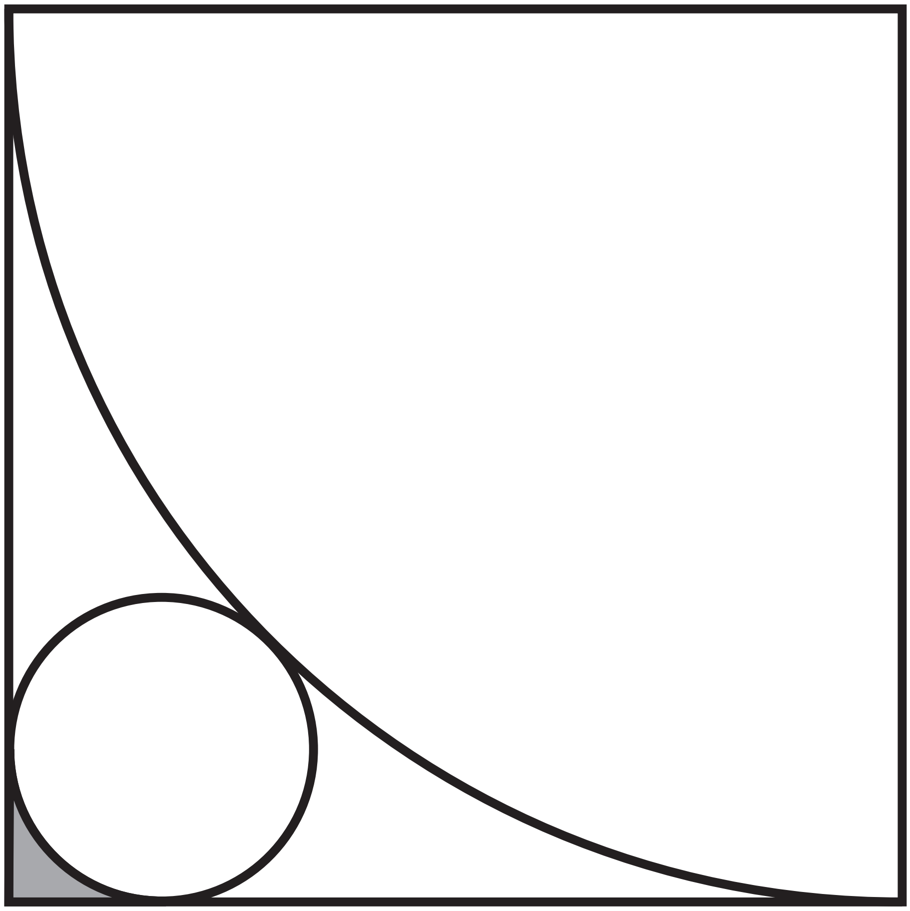
a) The system has exactly one solution (x, y) if and only if $|\lambda| \neq 1$.
b) The system has exactly one solution (x, y) for any parameter λ such that $|\lambda| > \sqrt{2}$.
c) For each solution (x, y) of the system we have $x = y$.
d) There is no solution of the system for $\lambda = -1$.
e) For $\lambda = 1$ there exists a solution (x, y) satisfying $x \geq 0, y \leq 0$.

9. At the upper left corner of an 8×8 chessboard there is a chess piece that needs to reach the lower right corner. With each move it can move either one square right or one square down. Denote by P the number of all possible paths of the chess piece.

a) $P \in [1600, 4024]$
b) $P \in [2048, 6500]$
c) $P \in [4000, 10256]$
d) P is odd.
e) P is divisible by three.

10. A circle with radius 1 is centered at a corner of a square with side length 1. Another circle touches both the boundary of the square and the circle. Calculate the area S of the grey region enclosed by the boundary of the smaller circle and the square (see the picture).

a) $S < (3 - 2\sqrt{2})^2$
b) $S = (1 - \frac{\pi}{4})(3 - 2\sqrt{2})^2$
c) $S = (1 - \frac{\pi}{4})(4 - 2\sqrt{2})^2$
d) $S > 1 - \frac{\pi}{4}$
e) $S > 10^{-6}$



Answers (A)

1. $M = [2, 3)$
Correct answers: c, d.

2. $M = \left(-\infty, -\frac{1}{2}(1+\sqrt{17})\right) \cup (3, \infty)$
Correct answers: b.

3. $d = 3$
Correct answers: a, d, e.

4. 432
Correct answers: b, c, d.

5. $M = \{\frac{\pi}{4} + 2k\pi;\ k \in \mathbb{Z}\} \cup \{\frac{3\pi}{4} + 2k\pi;\ k \in \mathbb{Z}\}$
Correct answers: b, c.

6. Correct answers: b, d, e.

7. Correct answers: a, d, e.

8. If $\lambda \neq \pm\sqrt{2}$, then the solution is $x = 1, y = 0$. If $\lambda = \pm\sqrt{2}$, then there is infinitely many solutions.
Correct answers: b, e.

9. $\binom{14}{7} = 3432$
Correct answers: a, b, e.

10. $S = (1 - \frac{\pi}{4})r^2$, kde $r = \frac{\sqrt{2}-1}{\sqrt{2}+1} = 3 - 2\sqrt{2}$.
Correct answers: a, b, e.