State Final Examination (Mathematics Sample Questions)

Spring 2023

1 Linear maps (3 points)

1. In the vector space $\mathbb{R}^3$, consider the linear map $f(x) = Ax$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 0 & 2 & 3 \end{pmatrix}.$$

   Find a line p satisfying the following two conditions:

   – p passes through the point $(1, 2, 3)^T$, and

   – the image of p under the linear map $f(x)$ is a single point.

2. Prove or disprove:

   A linear map $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism if and only if it maps any line onto a line.

2 Eigenvalues and eigenvectors (3 points)

1. Define the terms *eigenvalue and eigenvector* of a real matrix $A \in \mathbb{R}^{n\times n}$.
2. Formulate the spectral decomposition theorem for a real symmetric matrix $A \in \mathbb{R}^{n\times n}$.
3. Find a matrix $A \in \mathbb{R}^{3\times 3}$ such that the following conditions are satisfied:

   – the rank of A is 1, and

   – the eigenvectors of A are $(2, 1, 1)^T$, $(1, -1, -1)^T$ and $(0, 1, -1)^T$. Notice that the vectors are orthogonal.

3 Orthogonal complement (3 points)

1. Define the term *orthogonal complement* of a set of vectors.
2. Let V be a vector space of dimension n. Prove that a set of vectors $\{x_1, \ldots, x_m\} \subset V$ is linearly independent if and only if $\dim(\{x_1, \ldots, x_m\}^\perp) \leq n - m$.
3. Find a basis of the orthogonal complement of the solution set of the system of linear equations

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0,$$
$$2x_1 + x_2 + 2x_3 + x_4 = 0.$$

4 Inclusion–exclusion principle (3 points)

1. Write the statement of the Inclusion–exclusion principle.
2. How many integers from the set $\{1, 2, \ldots, 360\}$ is divisible by at least one of $4, 6, 9$?

5 Hypercube (3 points)

For $d \in \mathbb{N}$, let V_d be the set of all ordered d-tuples of zeros and ones, i.e., $V_d = \{0,1\}^d$. Let us define a graph H_d on the vertex set V_d, where two k-tuples are joined by an edge if and only if they differ in exactly one entry. In other words, we have $H_d = (V_d, E)$, with $E = \{\{x,y\} | x, y \in V_d \wedge \exists! i : x_i \neq y_i\}$. (For example, for $d = 4$, the vertex 1010 is adjacent to the vertices 0010, 1110, 1000, and 1011.) This graph is called a *d-dimensional hypercube*.

1. What is the maximum possible number of edges of a planar graph with n vertices?
2. For what values of d is the d-dimensional hypercube a planar graph? Justify your answer.

6 Sequences (3 points)

Let $(a_n) = (a_1, a_2, \ldots)$ be a sequence of real numbers.

1. Define what it means that (a_n) has the limit $+\infty$ (plus infinity).
2. Define what it means that (a_n) is unbounded from above.
3. Is it true that if the sequence (a_n) is unbounded from above then it has the limit $+\infty$? Justify your answer.

7 Continuous function (3 points)

Let $f : (0,1) \to \mathbb{R}$ be a real function.

1. Define what it means that f is continuous at the point $\frac{1}{2}$.
2. Is the function f defined on this interval by $f(x) = \frac{2x-1}{1-2x}$ for $x \neq \frac{1}{2}$ and by $f(\frac{1}{2}) = -1$ continuous at the point $\frac{1}{2}$?
3. In part 2, could not we define the function more simply by $f(x) = \frac{2x-1}{1-2x}$ for every $x \in (0,1)$?

Justify your answers in parts 2 and 3.

8 Riemann integral (3 points)

Let $f : [0,1] \to \mathbb{R}$ be a real function that has continuous derivative $f' : [0,1] \to \mathbb{R}$.

1. Compute the Riemann integral $\int_0^1 f'(x)\,\mathrm{d}x$.
2. Compute the Riemann integral $\int_0^1 x\mathrm{e}^{x^2}\,\mathrm{d}x$.
3. Compute the Riemann integral $\int_0^1 x\cos(x^2)\,\mathrm{d}x$.

Justify your answers (there is no need to justify common theorems and propositions from the lectures).

9 Skolemization and models (3 points)

Consider the following statements:

– "Every hare is faster than some tortoise."

– "Everyone is either a hare or a tortoise, but not both."

– "There exists at least one hare and at least one tortoise."

1. Express the above statements as *sentences* $\varphi_1, \varphi_2, \varphi_3$ in a suitable language L of the (first-order) predicate logic.
2. Using Skolemization, find an *open* theory S (in some extension L' of the language L) which is equisatisfiable with the theory $T = \{\varphi_1, \varphi_2, \varphi_3\}$. Give a *three-element* model of the theory S, or explain why it does not exist.
3. Additionally, consider the sentence φ_4 in the language L that formalizes the statement "There exists a hare that is faster than all tortoises." Is φ_4 true / contradictory / independent in the theory T? Justify your answer.