State Final Examination (Mathematics Sample Questions)

Fall 2022

1 Continuity and derivative (3 points)

1. Define continuity of a function at a point.
2. Let f be a function defined as

$$f(x) = \begin{cases} x \sin \frac{1}{x} \text{ for all } x \neq 0, \\ 0 \text{ for } x = 0. \end{cases}$$

   – At which points is the function f continuous? Justify your answer.

   – Calculate the derivative of the function f at all points in which it exists. (In particular, justify whether the derivative exists at 0 and if yes, determine its value.)

3. Let $a \in \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$. What implications hold between the following two statements? Justify your answer.

   P. Functions f and g are continuous at a.

   Q. The function $f + g$ is continuous at a.

2 Definite integral (3 points)

1. Write definitions of the upper and the lower Riemann sum, the upper and the lower Riemann integral and the Riemann integral.
2. Is the following function f Riemann integrable on the interval $[1, 2]$? Justify your answer.

$$f(x) = \begin{cases} x \text{ for all } x \in \mathbb{Q} \\ -x \text{ for all } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

3. Calculate the definite integral

$$\int_0^{\pi} x^2 \cos(2x) dx.$$

3 Linear transformation matrix (3 points)

Consider the linear transformation in plane $\mathbb{R}^2$, which maps the rectangle $[0, 2] \times [0, 1]$ onto the rectangle $[-2, 0] \times [-1, 0]$.

1. Find the matrix form of this linear transformation.
2. Determine the eigenvalues of this linear map (that is, the eigenvalues of the corresponding matrix).
3. Decide whether the vector $v = (2, 3)^T$ is an eigenvector of this linear map.

4 Surjective mappings (3 body)

1. Give a definition of an injective mapping and of a surjective mapping.
2. Consider a linear mapping $f\colon \mathbb{R}^3 \mapsto \mathbb{R}^3$ such that for every three vectors $x, y, z \in \mathbb{R}^3$ we have that their images $f(x), f(y), f(z)$ are linearly dependent. Prove that the mapping f is not surjective.

5 Inner product (3 points)

Consider the inner product in $\mathbb{R}^2$ given by

$$\langle x, y \rangle = 2x_1y_1 + x_2y_2 + x_1y_2 + x_2y_1.$$

1. For this inner product state the precise formulation of the Cauchy–Schwarz inequality.
2. For this inner product determine the matrix of the projection onto the line spanned by vector $(1,0)^T$.

6 Inclusion–exclusion principle (3 points)

1. State the inclusion–exclusion principle.
2. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$.
   (a) How many functions $f\colon [n] \to [n]$ have the property that for every even $x \in [n]$, we have $f(x) \neq x$?
   (b) How many injective functions $f\colon [n] \to [n]$ have the property that for every even $x \in [n]$, we have $f(x) \neq x$?

7 Spanning trees (3 points)

1. Define the term *spanning tree of a graph*.
2. Let $G = (V, E)$ be a connected graph. Suppose every edge $e \in E$ is assigned a weight $w(e) \in \mathbb{R}$, with no two distinct edges having the same weight. Let T be a minimum spanning tree of G, i.e., a spanning tree with the smallest possible sum of edge-weights. Show that for every edge $e \in E$, the following two statements are equivalent:
   (a) The edge e belongs to T.
   (b) Every cycle in the graph G that contains the edge e also contains at least one edge whose weight is greater than the weight of e.

8 Probability (3 points)

We repeatedly roll a regular dice (i.e., a six-sided dice labelled 1, …, 6). We denote by X the order of the roll when we first roll a six. Next, we denote by Y the number of sixes that came up on the first 100 rolls.

1. Determine $\mathbf{P}(X = 10)$. Name the distribution of random variable X.
2. Determine $\mathbf{P}(Y = 10)$. Name the distribution of random variable Y.
3. Determine the expectations $E(X)$, $E(Y)$, and $E(X + Y)$.

9 Logic (3 body)

1. Give definitions for predicate logic when a formula φ is *valid in a structure* $\mathcal{A}$, when φ is *valid in a theory* T, and when φ is *(logically) valid*.
2. Consider a theory $T = \{P(x,x),\ P(x,y) \to P(y,x),\ P(x,y) \to (P(y,z) \to P(x,z))\}$ of a language $L = \langle P \rangle$ without equality. Find a formula φ of the language L and a three-element model $\mathcal{A}$ of the theory T such that φ is valid in $\mathcal{A}$, but φ is not valid in T. Give an explanation why φ has the desired properties.
3. Find a formula ψ of the language L such that ψ is valid in T, but ψ is not (logically) valid. Give an explanation why ψ has the desired properties, including a formal proof of its validity in T using some proof system.