# On the algebraic structure of midpoint operations in C\*-algebras

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- 2 Midpoints, reflections and translations
- 3 The algebraic structure of the midpoint operations
- 4 Series expansion of midpoint operations

# Outline



- 2 Midpoints, reflections and translations
- 3 The algebraic structure of the midpoint operations
- 4 Series expansion of midpoint operations

# Notation

- $x^{T}$ ,  $A^{T}$  transpose of a (column) vector or matrix
- x\*, A\* conjugate transpose of a (complex) vector or matrix
- symmetric matrix:  $A^T = A$
- self-adjoint (Hermitian) matrix:  $A^* = A$
- orthogonal matrix:  $A^T = A^{-1}$
- unitary matrix:  $A^* = A^{-1}$

# **Spectral Decomposition Theorem**

- The eigenvalues are real.
- The eigenvectors are orthogonal.
- $A = Q \wedge Q^T$  with Q orthogonal. ( $A = U \wedge U^*$  with U unitary.)

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# Definition: Positive semi-definite matrix

### Definition: Positive definite matrix

- positive semi-definite
- $x^T A x = 0$  if and only if x = 0

#### Theorem

- it is symmetric/self-adjoint
- all eigenvalues are non-negative (positive)

# Definition: Positive semi-definite matrix

• real symmetric •  $x^T A x \ge 0$  for all vector x • complex self-adjoint •  $x^*Ax \ge 0$  for all vector x

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- exp(*A*)
- $\log(A)$  for ||A|| < 1
- $\exp(A + B) = \exp(A) \exp(B)$  if A, B commute
- exp(A) positive definite if and only if A is symmetric

- *A* + *B*, *AB*
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- Jordan product:  $A \circ B = \frac{1}{2}(AB + BA)$

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# **Operations on general matrices**

# Unary operations on general matrices

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### Real functions on symmetric matrices

Let f be a real valued function defined on the eigenvalues of A. Define

$$f(A) = Q \begin{bmatrix} f(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f(\lambda_n) \end{bmatrix} Q^T.$$

#### Unary operations on p.s.d. matrices

- log(A) is well defined if A is positive definite
- $A^t$  for  $t \ge 0$
- A<sup>1</sup>/<sub>2</sub>

### Binary operations on p.s.d. matrices

• ABA

• (1 - t)A + tB for  $0 \le t \le 1$ 

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### Definition

2-norm 
$$||A|| = \sup_{x \neq 0} \frac{||Ax||}{||x||}$$
  
Frobenius norm  $||A||_F = \sqrt{\text{Tr}(AA^*)}$ 

#### For p.s.d. matrices:

• 
$$||A|| = \lambda_{\max}$$
  
•  $||A||_F = \sqrt{\lambda_1^2 + \dots + \lambda_n^2}$ 

 $\bullet ||A|| \leq ||A||_F$ 

$$\langle A,B\rangle_F = \operatorname{Tr}(AB^*) = \sum_{i,j} a_{ij} \overline{b}_{ij}$$

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Associative algebra  $\mathcal{A}$  over  $\mathbb{C}$  with norm ||x|| such that

- $||xy|| \le ||x|| ||y||$
- A is complete in the metrix induced by ||.||

#### and an involution $x^*$ such that

- $(x + y)^* = x^* + y^*$ ,
- $(xy)^* = y^*x^*$ ,

$$(x^{-1})^* = (x^*)^{-1}$$
 if x is invertible

•  $||xx^*|| = ||x||^2$ 

#### Example: commutative C\*-algebras

C(X) for a compact Hausdorff space X.

#### <sup>-</sup>act

Self-adjoint, positive properties, exp(x), log(x),  $x^{\frac{1}{2}}$ , etc. can be defined.

 $(\lambda \mathbf{x})^* - \overline{\lambda} \mathbf{x}^*$ 

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and an involution  $x^*$  such that

- $(x + y)^* = x^* + y^*$ ,  $(\lambda x)^* = \overline{\lambda} x^*$
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#### Example: commutative C\*-algebras

C(X) for a compact Hausdorff space X.

#### <sup>-</sup>act

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- 2 Midpoints, reflections and translations
- 3 The algebraic structure of the midpoint operations
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#### Definition

• The Riemann metric of positive definite matrices A, B is

$$d_R(A,B) = \|\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\|$$

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Let (X, d) be a metric space, and  $p_1, \ldots, p_m$  points in X.

• The least square mean is the point  $x \in X$  which minimizes

$$d(x,p_1)^2+\cdots+d(x,p_m)^2.$$

• The weighted mean with weights  $0 \le w_1, \ldots, w_m$  minimizes

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- The midpoint of two points *p*, *q* is the least square mean of *p*, *q*.
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#### Example

## Least square means and midpoints

#### Definition

Let (X, d) be a metric space, and  $p_1, \ldots, p_m$  points in X.

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#### Example

In the Euclidean space, the (weighted) least square mean is the (weighted) arithmetic mean.

midpoint:  $x \# y = x^{\frac{1}{2}} \left( x^{-\frac{1}{2}} y x^{-\frac{1}{2}} \right)^{\frac{1}{2}} x^{\frac{1}{2}}$ weighted midpoint:  $x \#_t y = x^{\frac{1}{2}} \left( x^{-\frac{1}{2}} y x^{-\frac{1}{2}} \right)^t x^{\frac{1}{2}}$ 

$$x^{\frac{1}{2}} \left( x^{-\frac{1}{2}} y x^{-\frac{1}{2}} \right)^{\frac{1}{2}} x^{\frac{1}{2}} = z$$

$$x^{-\frac{1}{2}} y x^{-\frac{1}{2}} = \left( x^{-\frac{1}{2}} z x^{-\frac{1}{2}} \right)^{2} = x^{-\frac{1}{2}} z x^{-1} z x^{-\frac{1}{2}}$$

$$y = z x^{-1} z$$

- No square root is needed!
- **Translation** = product of two central reflections:  $x^{\frac{1}{2}}yx^{\frac{1}{2}}$ .

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$$x \diamond y = \frac{1}{4} x^{-\frac{1}{2}} \left( x + \sqrt{x^{\frac{1}{2}} y x^{\frac{1}{2}}} \right)^2 x^{-\frac{1}{2}}$$

Straightforward computations [Bhatia, Jain, Lim 2019] give that

$$x \diamond y = \frac{1}{4} \left( x + y + x (x^{-1} \# y) + (x^{-1} \# y) x \right).$$

and

$$(x(x^{-1}\#y))^2 = xy, \quad ((x^{-1}\#y)x)^2 = yx.$$

• With some cheating, we have

$$x \diamond y = \frac{1}{4} \left( x + y + (xy)^{\frac{1}{2}} + (yx)^{\frac{1}{2}} \right).$$

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- 1 Positive definite matrices
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### Using heavy C\*-algebra machinery, Lajos Molnár (Szeged) proved:

#### Theorem (Molnár 2023)

Let  $\mathcal{A}$  be a C\*-algebra and  $\mathcal{A}^{++}$  the set of positive definite elements. The bijective map  $\phi : \mathcal{A}^{++} \to \mathcal{A}^{++}$  preserves the Wasserstein mean if and only if there is

- a Jordan \*-automorphism  $J : \mathcal{A} \to \mathcal{A}$
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#### **Definition (Aschbacher)**

Let G be a group. The subset  $S \subseteq G$  is a twisted subgroup of G is

- $1 \in S$ ,
- $a^{-1} \in S$  for all  $a \in S$ ,
- $aba \in S$  for all  $a, b \in S$ .

#### Examples

- The set  $S = \{x \in G \mid \alpha(x) = x^{-1}\}$  of anti-fixed elements.
- The set  $T = \{x^{-1}\alpha(x) \mid x \in G\}$  of "commutators".
- Homework: Construct the set of positive definite matrices as set of commutators.

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#### Examples

- The set  $S = \{x \in G \mid \alpha(x) = x^{-1}\}$  of anti-fixed elements.
- The set  $T = \{x^{-1}\alpha(x) \mid x \in G\}$  of "commutators".
- Homework: Construct the set of positive definite matrices as set of commutators.

## Theorem [Glauberman 1968 (and Kiechle, and Ungar, and ...)]

• The reflection



is well-defined in any twisted subgroup S.

- It is a left quasigroup.
- Moreover, it is a quasigroup if and only if *S* is 2-divisible.
- In the latter case, its loop isotopy is given by the operation  $x^{\frac{1}{2}}yx^{\frac{1}{2}}$ .

#### (A version of) Glauberman's Z\*-theorem

If *S* is a finite twisted subgroup of odd order, then  $\langle S \rangle$  has odd order. In particular, the loop  $x^{\frac{1}{2}}yx^{\frac{1}{2}}$  is solvable.

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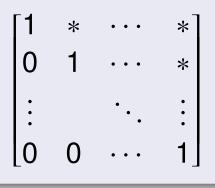
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## (Upper) unipotent matrix

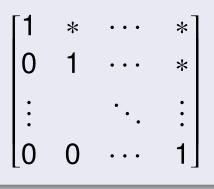


#### Observations

- Convex combination of unipotent matrices are unipotent.
- $A^n = 1$  for  $n \times n$  unipotent A.
- Inverse and square root are well-defined for unipotent matrices.
- So is the Wasserstein mean

$$x \diamond y = \frac{1}{4} x^{-\frac{1}{2}} \left( x + \sqrt{x^{\frac{1}{2}} y x^{\frac{1}{2}}} \right)^2 x^{-\frac{1}{2}}.$$

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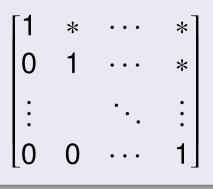


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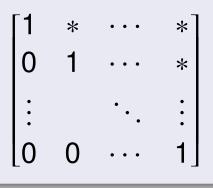


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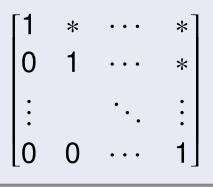
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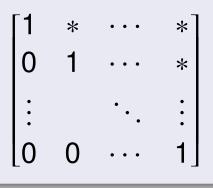
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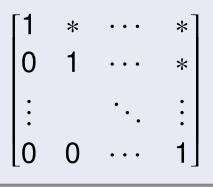
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Let *B* be a unitary associative algebra over  $\mathbb{C}$ . We say that *B* is a split *C*\*-by-nilpotent algebra, if the following conditions hold:

- If  $\mathbb{O}$  B has a unitary subalgebra  $\mathcal{A}$  that is a C\*-algebra and the unit of  $\mathcal{A}$  coincides with the unit of B;
- B has an ideal N which consists of nilpotent elements;
- $B = \mathcal{A} + N;$
- The elements of *A* and *N* commute.

#### emma

$$B^{++} = \{a + n \mid a \in \mathcal{A}^{++}, n \in N\}.$$

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#### Lemma

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- [[x, y], z] = 0 holds if *N* is 2-step nilpotent.
- This gives non-commutative split C\*-by-nilpotent algebra with associative 4(x ◊ y).
- In a C\*-algebra, [[x, y], z] = 0 implies [x, y] = 0, hence commutativity.
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- Positive definite matrices
- 2 Midpoints, reflections and translations
- 3 The algebraic structure of the midpoint operations
- 4 Series expansion of midpoint operations

# Lie triple systems

 Lie triple systems are ternary operations [x, y, z] motivated by the double Lie bracket

[[x, y], z].

- In a Lie group, twisted subgroups correspond to Lie triple subsystems of the tangent Lie algebra.
- If (L, ●) is a real analytic Bruck loop, then its tangent space is an (abstract) Lie triple system.
- Baker-Campbell-Hausdorff formula (GN 2002): Near the identity, the Bruck loop operation can be expressed as an infinite series of L.t.s. terms:

$$\log(\exp(x) \bullet \exp(y)) = x + y + \frac{1}{3}[y, x, x] - \frac{1}{6}[x, y, y] + \cdots$$

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# THANK YOU FOR YOUR ATTENTION!