On the algebraic structure of midpoint operations in C*-algebras

Gábor P. Nagy
joint work with G. La Rosa and M. Mancini

University of Szeged (Hungary) and
HUN-REN-ELTE Geometric and Algebraic Combinatorics Research Group

Algebra Colloquium
April 23, 2024
Charles University (Prague)

Outline

Outline

Symmetric (self-adjoint) matrices

Notation

- x^T, A^T transpose of a (column) vector or matrix
- x^*, A^* conjugate transpose of a (complex) vector or matrix
- symmetric matrix: $A^T = A$
- self-adjoint (Hermitian) matrix: $A^* = A$
- orthogonal matrix: $A^T = A^{-1}$
- unitary matrix: $A^* = A^{-1}$

Spectral Decomposition Theorem

Let A be a real symmetric (complex self-adjoint) matrix.

- The eigenvalues are real.
- The eigenvectors are orthogonal.
- $A = Q \Lambda Q^T$ with Q orthogonal. ($A = U \Lambda U^*$ with U unitary.)

Symmetric (self-adjoint) matrices

Notation

- x^T, A^T transpose of a (column) vector or matrix
- x^*, A^* conjugate transpose of a (complex) vector or matrix
- symmetric matrix: $A^T = A$
- self-adjoint (Hermitian) matrix: $A^* = A$
- orthogonal matrix: $A^T = A^{-1}$
- unitary matrix: $A^* = A^{-1}$

Spectral Decomposition Theorem

Let A be a real symmetric (complex self-adjoint) matrix.

- The eigenvalues are real.
- The eigenvectors are orthogonal.
- $A = Q\Lambda Q^T$ with Q orthogonal. ($A = U\Lambda U^*$ with U unitary.)

Symmetric (self-adjoint) matrices

Notation

- x^T, A^T transpose of a (column) vector or matrix
- x^*, A^* conjugate transpose of a (complex) vector or matrix
- symmetric matrix: $A^T = A$
- self-adjoint (Hermitian) matrix: $A^* = A$
- orthogonal matrix: $A^T = A^{-1}$
- unitary matrix: $A^* = A^{-1}$

Spectral Decomposition Theorem

Let A be a real symmetric (complex self-adjoint) matrix.

- The eigenvalues are real.
- The eigenvectors are orthogonal.
- $A = Q \Lambda Q^T$ with Q orthogonal. ($A = U \Lambda U^*$ with U unitary.)

Symmetric (self-adjoint) matrices

Notation

- x^T, A^T transpose of a (column) vector or matrix
- x^*, A^* conjugate transpose of a (complex) vector or matrix
- symmetric matrix: $A^T = A$
- self-adjoint (Hermitian) matrix: $A^* = A$
- orthogonal matrix: $A^T = A^{-1}$
- unitary matrix: $A^* = A^{-1}$

Spectral Decomposition Theorem

Let A be a real symmetric (complex self-adjoint) matrix.

- The eigenvalues are real.
- The eigenvectors are orthogonal.
- $A = Q\Lambda Q^T$ with Q orthogonal. ($A = U\Lambda U^*$ with U unitary.)

Symmetric (self-adjoint) matrices

Notation

- x^T, A^T transpose of a (column) vector or matrix
- x^*, A^* conjugate transpose of a (complex) vector or matrix
- symmetric matrix: $A^T = A$
- self-adjoint (Hermitian) matrix: $A^* = A$
- orthogonal matrix: $A^T = A^{-1}$
- unitary matrix: $A^* = A^{-1}$

Spectral Decomposition Theorem

Let A be a real symmetric (complex self-adjoint) matrix.

- The eigenvalues are real.
- The eigenvectors are orthogonal.
- $A = Q\Lambda Q^T$ with Q orthogonal. ($A = U\Lambda U^*$ with U unitary.)

Symmetric (self-adjoint) matrices

Notation

- x^T, A^T transpose of a (column) vector or matrix
- x^*, A^* conjugate transpose of a (complex) vector or matrix
- symmetric matrix: $A^T = A$
- self-adjoint (Hermitian) matrix: $A^* = A$
- orthogonal matrix: $A^T = A^{-1}$
- unitary matrix: $A^* = A^{-1}$

Spectral Decomposition Theorem

Let A be a real symmetric (complex self-adjoint) matrix.

- The eigenvalues are real.
- The eigenvectors are orthogonal.
- $A = Q\Lambda Q^T$ with Q orthogonal. ($A = U\Lambda U^*$ with U unitary.)

Symmetric (self-adjoint) matrices

Notation

- x^T, A^T transpose of a (column) vector or matrix
- x^*, A^* conjugate transpose of a (complex) vector or matrix
- symmetric matrix: $A^T = A$
- self-adjoint (Hermitian) matrix: $A^* = A$
- orthogonal matrix: $A^T = A^{-1}$
- unitary matrix: $A^* = A^{-1}$

Spectral Decomposition Theorem

Let A be a real symmetric (complex self-adjoint) matrix.

- The eigenvalues are real.
- The eigenvectors are orthogonal.
- $A = Q\Lambda Q^T$ with Q orthogonal. ($A = U\Lambda U^*$ with U unitary.)

Symmetric (self-adjoint) matrices

Notation

- x^T, A^T transpose of a (column) vector or matrix
- x^*, A^* conjugate transpose of a (complex) vector or matrix
- symmetric matrix: $A^T = A$
- self-adjoint (Hermitian) matrix: $A^* = A$
- orthogonal matrix: $A^T = A^{-1}$
- unitary matrix: $A^* = A^{-1}$

Spectral Decomposition Theorem

Let A be a real symmetric (complex self-adjoint) matrix.

- The eigenvalues are real.
- The eigenvectors are orthogonal.
- $A = Q \Lambda Q^T$ with Q orthogonal. ($A = U \Lambda U^*$ with U unitary.)

Symmetric (self-adjoint) matrices

Notation

- x^T, A^T transpose of a (column) vector or matrix
- x^*, A^* conjugate transpose of a (complex) vector or matrix
- symmetric matrix: $A^T = A$
- self-adjoint (Hermitian) matrix: $A^* = A$
- orthogonal matrix: $A^T = A^{-1}$
- unitary matrix: $A^* = A^{-1}$

Spectral Decomposition Theorem

Let A be a real symmetric (complex self-adjoint) matrix.

- The eigenvalues are real.
- The eigenvectors are orthogonal.
- $A = Q \Lambda Q^T$ with Q orthogonal. ($A = U \Lambda U^*$ with U unitary.)

Symmetric (self-adjoint) matrices

Notation

- x^T, A^T transpose of a (column) vector or matrix
- x^*, A^* conjugate transpose of a (complex) vector or matrix
- symmetric matrix: $A^T = A$
- self-adjoint (Hermitian) matrix: $A^* = A$
- orthogonal matrix: $A^T = A^{-1}$
- unitary matrix: $A^* = A^{-1}$

Spectral Decomposition Theorem

Let A be a real symmetric (complex self-adjoint) matrix.

- The eigenvalues are real.
- The eigenvectors are orthogonal.
- $A = Q\Lambda Q^T$ with Q orthogonal. ($A = U\Lambda U^*$ with U unitary.)

Symmetric (self-adjoint) matrices

Notation

- x^T, A^T transpose of a (column) vector or matrix
- x^*, A^* conjugate transpose of a (complex) vector or matrix
- symmetric matrix: $A^T = A$
- self-adjoint (Hermitian) matrix: $A^* = A$
- orthogonal matrix: $A^T = A^{-1}$
- unitary matrix: $A^* = A^{-1}$

Spectral Decomposition Theorem

Let A be a real symmetric (complex self-adjoint) matrix.

- The eigenvalues are real.
- The eigenvectors are orthogonal.
- $A = Q\Lambda Q^T$ with Q orthogonal. ($A = U\Lambda U^*$ with U unitary.)

Positive semi-definite (p.s.d.) matrices

Definition: Positive semi-definite matrix

Definition: Positive definite matrix

- positive semi-definite
- $x^T A x = 0$ if and only if $x = 0$

Theorem

A matrix is positive semi-definite (positive definite) if and only if

- it is symmetric/self-adjoint
- all eigenvalues are non-negative (positive)

Positive semi-definite (p.s.d.) matrices

Definition: Positive semi-definite matrix

- real symmetric
- $x^T Ax \geq 0$ for all vector x

- complex self-adjoint
- $x^* Ax \geq 0$ for all vector x

Definition: Positive definite matrix

- positive semi-definite
- $x^T Ax = 0$ if and only if $x = 0$

Theorem

A matrix is positive semi-definite (positive definite) if and only if

- it is symmetric/self-adjoint
- all eigenvalues are non-negative (positive)

Positive semi-definite (p.s.d.) matrices

Definition: Positive semi-definite matrix

- real symmetric
- $x^T Ax \geq 0$ for all vector x

- complex self-adjoint
- $x^* Ax \geq 0$ for all vector x

Definition: Positive definite matrix

- positive semi-definite
- $x^T Ax = 0$ if and only if $x = 0$

Theorem

A matrix is positive semi-definite (positive definite) if and only if

- it is symmetric/self-adjoint
- all eigenvalues are non-negative (positive)

Positive semi-definite (p.s.d.) matrices

Definition: Positive semi-definite matrix

- real symmetric
- $x^T Ax \geq 0$ for all vector x

- complex self-adjoint
- $x^* Ax \geq 0$ for all vector x

Definition: Positive definite matrix

- positive semi-definite
- $x^T Ax = 0$ if and only if $x = 0$

Theorem

A matrix is positive semi-definite (positive definite) if and only if

- it is symmetric/self-adjoint
- all eigenvalues are non-negative (positive)

Positive semi-definite (p.s.d.) matrices

Definition: Positive semi-definite matrix

- real symmetric
- $x^T Ax \geq 0$ for all vector x

- complex self-adjoint
- $x^* Ax \geq 0$ for all vector x

Definition: Positive definite matrix

- positive semi-definite
- $x^T Ax = 0$ if and only if $x = 0$

Theorem

A matrix is positive semi-definite (positive definite) if and only if

- it is symmetric/self-adjoint
- all eigenvalues are non-negative (positive)

Positive semi-definite (p.s.d.) matrices

Definition: Positive semi-definite matrix

- real symmetric
- $x^T Ax \geq 0$ for all vector x

- complex self-adjoint
- $x^* Ax \geq 0$ for all vector x

Definition: Positive definite matrix

- positive semi-definite
- $x^T Ax = 0$ if and only if $x = 0$

Theorem

A matrix is positive semi-definite (positive definite) if and only if

- it is symmetric/self-adjoint
- all eigenvalues are non-negative (positive)

Positive semi-definite (p.s.d.) matrices

Definition: Positive semi-definite matrix

- real symmetric
- $x^T Ax \geq 0$ for all vector x
- complex self-adjoint
- $x^* Ax \geq 0$ for all vector x

Definition: Positive definite matrix

- positive semi-definite
- $x^T Ax = 0$ if and only if $x = 0$

Theorem

A matrix is positive semi-definite (positive definite) if and only if

- it is symmetric/self-adjoint
- all eigenvalues are non-negative (positive)

Positive semi-definite (p.s.d.) matrices

Definition: Positive semi-definite matrix

- real symmetric
- $x^T A x \geq 0$ for all vector x

- complex self-adjoint
- $x^* A x \geq 0$ for all vector x

Definition: Positive definite matrix

- positive semi-definite
- $x^T A x = 0$ if and only if $x = 0$

Theorem

A matrix is positive semi-definite (positive definite) if and only if

- it is symmetric/self-adjoint
- all eigenvalues are non-negative (positive)

Positive semi-definite (p.s.d.) matrices

Definition: Positive semi-definite matrix

- real symmetric
- $x^T Ax \geq 0$ for all vector x
- complex self-adjoint
- $x^* Ax \geq 0$ for all vector x

Definition: Positive definite matrix

- positive semi-definite
- $x^T Ax = 0$ if and only if $x = 0$

Theorem

A matrix is positive semi-definite (positive definite) if and only if

- it is symmetric/self-adjoint
- all eigenvalues are non-negative (positive)

Positive semi-definite (p.s.d.) matrices

Definition: Positive semi-definite matrix

- real symmetric
- $x^T A x \geq 0$ for all vector x

- complex self-adjoint
- $x^* A x \geq 0$ for all vector x

Definition: Positive definite matrix

- positive semi-definite
- $x^T A x = 0$ if and only if $x = 0$

Theorem

A matrix is positive semi-definite (positive definite) if and only if

- it is symmetric/self-adjoint
- all eigenvalues are non-negative (positive)

Positive semi-definite (p.s.d.) matrices

Definition: Positive semi-definite matrix

- real symmetric
- $x^T A x \geq 0$ for all vector x
- complex self-adjoint
- $x^* A x \geq 0$ for all vector x

Definition: Positive definite matrix

- positive semi-definite
- $x^T A x = 0$ if and only if $x = 0$

Theorem

A matrix is positive semi-definite (positive definite) if and only if

- it is symmetric/self-adjoint
- all eigenvalues are non-negative (positive)

Positive semi-definite (p.s.d.) matrices

Definition: Positive semi-definite matrix

- real symmetric
- $x^T Ax \geq 0$ for all vector x
- complex self-adjoint
- $x^* Ax \geq 0$ for all vector x

Definition: Positive definite matrix

- positive semi-definite
- $x^T Ax = 0$ if and only if $x = 0$

Theorem

A matrix is positive semi-definite (positive definite) if and only if

- it is symmetric/self-adjoint
- all eigenvalues are non-negative (positive)

Positive semi-definite (p.s.d.) matrices

Definition: Positive semi-definite matrix

- real symmetric
- $x^T A x \geq 0$ for all vector x
- complex self-adjoint
- $x^* A x \geq 0$ for all vector x

Definition: Positive definite matrix

- positive semi-definite
- $x^T A x = 0$ if and only if $x = 0$

Theorem

A matrix is positive semi-definite (positive definite) if and only if

- it is symmetric/self-adjoint
- all eigenvalues are non-negative (positive)

Positive semi-definite (p.s.d.) matrices

Definition: Positive semi-definite matrix

- real symmetric
- $x^T A x \geq 0$ for all vector x

- complex self-adjoint
- $x^* A x \geq 0$ for all vector x

Definition: Positive definite matrix

- positive semi-definite
- $x^T A x = 0$ if and only if $x = 0$

Theorem

A matrix is positive semi-definite (positive definite) if and only if

- it is symmetric/self-adjoint
- all eigenvalues are non-negative (positive)

Operations on general matrices

Unary operations on general matrices

- $\exp(A)$
- $\log(A)$ for $\|A\| < 1$
- $\exp(A + B) = \exp(A)\exp(B)$ if A, B commute
- $\exp(A)$ positive definite if and only if A is symmetric

Binary operations on general matrices

- $A + B$, AB
- $\exp(A + B) = \exp(A)\exp(B)$ if A, B commute
- Lie bracket: $[A, B] = AB - BA$
- Jordan product: $A \circ B = \frac{1}{2}(AB + BA)$

Operations on general matrices

Unary operations on general matrices

- $\exp(A)$
- $\log(A)$ for $\|A\| < 1$
- $\exp(A + B) = \exp(A)\exp(B)$ if A, B commute
- $\exp(A)$ positive definite if and only if A is symmetric

Binary operations on general matrices

- $A + B$, AB
- $\exp(A + B) = \exp(A)\exp(B)$ if A, B commute
- Lie bracket: $[A, B] = AB - BA$
- Jordan product: $A \circ B = \frac{1}{2}(AB + BA)$

Operations on general matrices

Unary operations on general matrices

- $\exp(A)$
- $\log(A)$ for $\|A\| < 1$
- $\exp(A+B) = \exp(A)\exp(B)$ if A, B commute
- $\exp(A)$ positive definite if and only if A is symmetric

Binary operations on general matrices

- $A + B$, AB
- $\exp(A+B) = \exp(A)\exp(B)$ if A, B commute
- Lie bracket: $[A, B] = AB - BA$
- Jordan product: $A \circ B = \frac{1}{2}(AB + BA)$

Operations on general matrices

Unary operations on general matrices

- $\exp(A)$
- $\log(A)$ for $\|A\| < 1$
- $\exp(A + B) = \exp(A)\exp(B)$ if A, B commute
- $\exp(A)$ positive definite if and only if A is symmetric

Binary operations on general matrices

- $A + B$, AB
- $\exp(A + B) = \exp(A)\exp(B)$ if A, B commute
- Lie bracket: $[A, B] = AB - BA$
- Jordan product: $A \circ B = \frac{1}{2}(AB + BA)$

Operations on general matrices

Unary operations on general matrices

- $\exp(A)$
- $\log(A)$ for $\|A\| < 1$
- $\exp(A+B) = \exp(A)\exp(B)$ if A, B commute
- $\exp(A)$ positive definite if and only if A is symmetric

Binary operations on general matrices

- $A+B$, AB
- $\exp(A+B) = \exp(A)\exp(B)$ if A, B commute
- Lie bracket: $[A, B] = AB - BA$
- Jordan product: $A \circ B = \frac{1}{2}(AB + BA)$

Operations on general matrices

Unary operations on general matrices

- $\exp(A)$
- $\log(A)$ for $\|A\| < 1$
- $\exp(A+B) = \exp(A)\exp(B)$ if A, B commute
- $\exp(A)$ positive definite if and only if A is symmetric

Binary operations on general matrices

- $A + B$, AB
- $\exp(A+B) = \exp(A)\exp(B)$ if A, B commute
- Lie bracket: $[A, B] = AB - BA$
- Jordan product: $A \circ B = \frac{1}{2}(AB + BA)$

Operations on general matrices

Unary operations on general matrices

- $\exp(A)$
- $\log(A)$ for $\|A\| < 1$
- $\exp(A + B) = \exp(A)\exp(B)$ if A, B commute
- $\exp(A)$ positive definite if and only if A is symmetric

Binary operations on general matrices

- $A + B$, AB
- $\exp(A + B) = \exp(A)\exp(B)$ if A, B commute
- Lie bracket: $[A, B] = AB - BA$
- Jordan product: $A \circ B = \frac{1}{2}(AB + BA)$

Operations on general matrices

Unary operations on general matrices

- $\exp(A)$
- $\log(A)$ for $\|A\| < 1$
- $\exp(A+B) = \exp(A)\exp(B)$ if A, B commute
- $\exp(A)$ positive definite if and only if A is symmetric

Binary operations on general matrices

- $A + B$, AB
- $\exp(A+B) = \exp(A)\exp(B)$ if A, B commute
- Lie bracket: $[A, B] = AB - BA$
- Jordan product: $A \circ B = \frac{1}{2}(AB + BA)$

Operations on general matrices

Unary operations on general matrices

- $\exp(A)$
- $\log(A)$ for $\|A\| < 1$
- $\exp(A + B) = \exp(A)\exp(B)$ if A, B commute
- $\exp(A)$ positive definite if and only if A is symmetric

Binary operations on general matrices

- $A + B$, AB
- $\exp(A + B) = \exp(A)\exp(B)$ if A, B commute
- Lie bracket: $[A, B] = AB - BA$
- Jordan product: $A \circ B = \frac{1}{2}(AB + BA)$

Operations on general matrices

Unary operations on general matrices

- $\exp(A)$
- $\log(A)$ for $\|A\| < 1$
- $\exp(A + B) = \exp(A)\exp(B)$ if A, B commute
- $\exp(A)$ positive definite if and only if A is symmetric

Binary operations on general matrices

- $A + B$, AB
- $\exp(A + B) = \exp(A)\exp(B)$ if A, B commute
- Lie bracket: $[A, B] = AB - BA$
- Jordan product: $A \circ B = \frac{1}{2}(AB + BA)$

Operations on p.s.d. matrices

Real functions on symmetric matrices

Let f be a real valued function defined on the eigenvalues of A. Define

$$f(A) = Q \begin{bmatrix} f(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f(\lambda_n)) \end{bmatrix} Q^T.$$

Unary operations on p.s.d. matrices

- $\log(A)$ is well defined if A is positive definite
- A^t for $t \geq 0$
- $A^{\frac{1}{2}}$

Binary operations on p.s.d. matrices

- ABA
- $(1-t)A + tB$ for $0 \leq t \leq 1$

Operations on p.s.d. matrices

Real functions on symmetric matrices

Let f be a real valued function defined on the eigenvalues of A. Define

$$f(A) = Q \begin{bmatrix} f(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f(\lambda_n)) \end{bmatrix} Q^T.$$

Unary operations on p.s.d. matrices

- $\log(A)$ is well defined if A is positive definite
- A^t for $t \geq 0$
- $A^{\frac{1}{2}}$

Binary operations on p.s.d. matrices

- ABA
- $(1-t)A + tB$ for $0 \leq t \leq 1$

Operations on p.s.d. matrices

Real functions on symmetric matrices

Let f be a real valued function defined on the eigenvalues of A. Define

$$f(A) = Q \begin{bmatrix} f(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f(\lambda_n)) \end{bmatrix} Q^T.$$

Unary operations on p.s.d. matrices

- $\log(A)$ is well defined if A is positive definite
- A^t for $t \geq 0$
- $A^{\frac{1}{2}}$

Binary operations on p.s.d. matrices

- ABA
- $(1-t)A + tB$ for $0 \leq t \leq 1$

Operations on p.s.d. matrices

Real functions on symmetric matrices

Let f be a real valued function defined on the eigenvalues of A. Define

$$f(A) = Q \begin{bmatrix} f(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f(\lambda_n)) \end{bmatrix} Q^T.$$

Unary operations on p.s.d. matrices

- $\log(A)$ is well defined if A is positive definite
- A^t for $t \geq 0$
- $A^{\frac{1}{2}}$

Binary operations on p.s.d. matrices

- ABA
- $(1-t)A + tB$ for $0 \leq t \leq 1$

Operations on p.s.d. matrices

Real functions on symmetric matrices

Let f be a real valued function defined on the eigenvalues of A. Define

$$f(A) = Q \begin{bmatrix} f(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f(\lambda_n)) \end{bmatrix} Q^T.$$

Unary operations on p.s.d. matrices

- $\log(A)$ is well defined if A is positive definite
- A^t for $t \geq 0$
- $A^{\frac{1}{2}}$

Binary operations on p.s.d. matrices

- ABA
- $(1-t)A + tB$ for $0 \leq t \leq 1$

Operations on p.s.d. matrices

Real functions on symmetric matrices

Let f be a real valued function defined on the eigenvalues of A. Define

$$f(A) = Q \begin{bmatrix} f(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f(\lambda_n)) \end{bmatrix} Q^T.$$

Unary operations on p.s.d. matrices

- $\log(A)$ is well defined if A is positive definite
- A^t for $t \geq 0$
- $A^{\frac{1}{2}}$

Binary operations on p.s.d. matrices

- ABA
- $(1-t)A + tB$ for $0 \leq t \leq 1$

Operations on p.s.d. matrices

Real functions on symmetric matrices

Let f be a real valued function defined on the eigenvalues of A. Define

$$f(A) = Q \begin{bmatrix} f(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f(\lambda_n)) \end{bmatrix} Q^T.$$

Unary operations on p.s.d. matrices

- $\log(A)$ is well defined if A is positive definite
- A^t for $t \geq 0$
- $A^{\frac{1}{2}}$

Binary operations on p.s.d. matrices

- ABA
- $(1-t)A + tB$ for $0 \leq t \leq 1$

Operations on p.s.d. matrices

Real functions on symmetric matrices

Let f be a real valued function defined on the eigenvalues of A. Define

$$f(A) = Q \begin{bmatrix} f(\lambda_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & f(\lambda_n)) \end{bmatrix} Q^T.$$

Unary operations on p.s.d. matrices

- $\log(A)$ is well defined if A is positive definite
- A^t for $t \geq 0$
- $A^{\frac{1}{2}}$

Binary operations on p.s.d. matrices

- ABA
- $(1-t)A + tB$ for $0 \leq t \leq 1$

Matrix norms

Definition

$$\text{2-norm} \quad ||A|| = \sup_{x \neq 0} \frac{||Ax||}{||x||}$$

$$\text{Frobenius norm} \quad ||A||_F = \sqrt{\mathrm{Tr}(AA^*)}$$

For p.s.d. matrices:

- $||A|| = \lambda_{\max}$
- $||A||_F = \sqrt{\lambda_1^2 + \cdots + \lambda_n^2}$
- $||A|| \leq ||A||_F$

Frobenius inner product

$$\langle A, B \rangle_F = \mathrm{Tr}(AB^*) = \sum_{i,j} a_{ij}\bar{b}_{ij}$$

Matrix norms

Definition

$$\text{2-norm} \quad ||A|| = \sup_{x \neq 0} \frac{||Ax||}{||x||}$$

$$\text{Frobenius norm} \quad ||A||_F = \sqrt{\text{Tr}(AA^*)}$$

For p.s.d. matrices:

- $||A|| = \lambda_{\text{max}}$
- $||A||_F = \sqrt{\lambda_1^2 + \cdots + \lambda_n^2}$
- $||A|| \leq ||A||_F$

Frobenius inner product

$$\langle A, B \rangle_F = \text{Tr}(AB^*) = \sum_{i,j} a_{ij} \bar{b}_{ij}$$

Matrix norms

Definition

$$\text{2-norm} \quad ||A|| = \sup_{x \neq 0} \frac{||Ax||}{||x||}$$

$$\text{Frobenius norm} \quad ||A||_F = \sqrt{\mathrm{Tr}(AA^*)}$$

For p.s.d. matrices:

- $||A|| = \lambda_{\max}$
- $||A||_F = \sqrt{\lambda_1^2 + \cdots + \lambda_n^2}$
- $||A|| \leq ||A||_F$

Frobenius inner product

$$\langle A, B \rangle_F = \mathrm{Tr}(AB^*) = \sum_{i,j} a_{ij} \bar{b}_{ij}$$

Matrix norms

Definition

$$\text{2-norm} \quad ||A|| = \sup_{x \neq 0} \frac{||Ax||}{||x||}$$

$$\text{Frobenius norm} \quad ||A||_F = \sqrt{\mathrm{Tr}(AA^*)}$$

For p.s.d. matrices:

- $||A|| = \lambda_{\max}$
- $||A||_F = \sqrt{\lambda_1^2 + \cdots + \lambda_n^2}$
- $||A|| \leq ||A||_F$

Frobenius inner product

$$\langle A, B \rangle_F = \mathrm{Tr}(AB^*) = \sum_{i,j} a_{ij} \bar{b}_{ij}$$

Matrix norms

Definition

$$\text{2-norm} \quad ||A|| = \sup_{x \neq 0} \frac{||Ax||}{||x||}$$

$$\text{Frobenius norm} \quad ||A||_F = \sqrt{\mathrm{Tr}(AA^*)}$$

For p.s.d. matrices:

- $||A|| = \lambda_{\max}$
- $||A||_F = \sqrt{\lambda_1^2 + \cdots + \lambda_n^2}$
- $||A|| \leq ||A||_F$

Frobenius inner product

$$\langle A, B \rangle_F = \mathrm{Tr}(AB^*) = \sum_{i,j} a_{ij} \bar{b}_{ij}$$

C*-algebras

Definition

Associative algebra $\mathcal{A}$ over $\mathbb{C}$ with norm $||x||$ such that

- $||xy|| \leq ||x|| \, ||y||$
- A is complete in the metrix induced by $||.||$

and an involution x^* such that

- $(x+y)^* = x^* + y^*$, $(\lambda x)^* = \bar{\lambda} x^*$
- $(xy)^* = y^* x^*$, $(x^{-1})^* = (x^*)^{-1}$ if x is invertible
- $||xx^*|| = ||x||^2$

Example: commutative C*-algebras

$C(X)$ for a compact Hausdorff space X.

Fact

Self-adjoint, positive properties, $\exp(x)$, $\log(x)$, $x^{\frac{1}{2}}$, etc. can be defined.

C*-algebras

Definition

Associative algebra $\mathcal{A}$ over $\mathbb{C}$ with norm $||x||$ such that

- $||xy|| \leq ||x|| \, ||y||$
- A is complete in the metrix induced by $||.||$

and an involution x^* such that

- $(x+y)^* = x^* + y^*$, $(\lambda x)^* = \bar{\lambda} x^*$
- $(xy)^* = y^* x^*$, $(x^{-1})^* = (x^*)^{-1}$ if x is invertible
- $||xx^*|| = ||x||^2$

Example: commutative C*-algebras

$C(X)$ for a compact Hausdorff space X.

Fact

Self-adjoint, positive properties, $\exp(x)$, $\log(x)$, $x^{\frac{1}{2}}$, etc. can be defined.

C*-algebras

Definition

Associative algebra $\mathcal{A}$ over $\mathbb{C}$ with norm $||x||$ such that

- $||xy|| \leq ||x|| \, ||y||$
- A is complete in the metrix induced by $||.||$

and an involution x^* such that

- $(x+y)^* = x^* + y^*$, $\qquad (\lambda x)^* = \bar{\lambda} x^*$
- $(xy)^* = y^* x^*$, $\qquad (x^{-1})^* = (x^*)^{-1}$ if x is invertible
- $||xx^*|| = ||x||^2$

Example: commutative C*-algebras

$C(X)$ for a compact Hausdorff space X.

Fact

Self-adjoint, positive properties, $\exp(x)$, $\log(x)$, $x^{\frac{1}{2}}$, etc. can be defined.

C*-algebras

Definition

Associative algebra $\mathcal{A}$ over $\mathbb{C}$ with norm $||x||$ such that

- $||xy|| \leq ||x||\,||y||$
- A is complete in the metrix induced by $||.||$

and an involution x^* such that

- $(x+y)^* = x^* + y^*$, $\quad$ $(\lambda x)^* = \bar{\lambda} x^*$
- $(xy)^* = y^* x^*$, $\quad$ $(x^{-1})^* = (x^*)^{-1}$ if x is invertible
- $||xx^*|| = ||x||^2$

Example: commutative C*-algebras

$C(X)$ for a compact Hausdorff space X.

Fact

Self-adjoint, positive properties, $\exp(x)$, $\log(x)$, $x^{\frac{1}{2}}$, etc. can be defined.

C*-algebras

Definition

Associative algebra $\mathcal{A}$ over $\mathbb{C}$ with norm $||x||$ such that

- $||xy|| \leq ||x|| \, ||y||$
- A is complete in the metrix induced by $||.||$

and an involution x^* such that

- $(x+y)^* = x^* + y^*$, $\quad (\lambda x)^* = \bar{\lambda} x^*$
- $(xy)^* = y^* x^*$, $\quad (x^{-1})^* = (x^*)^{-1}$ if x is invertible
- $||xx^*|| = ||x||^2$

Example: commutative C*-algebras

$C(X)$ for a compact Hausdorff space X.

Fact

Self-adjoint, positive properties, $\exp(x)$, $\log(x)$, $x^{\frac{1}{2}}$, etc. can be defined.

C*-algebras

Definition

Associative algebra $\mathcal{A}$ over $\mathbb{C}$ with norm $||x||$ such that

- $||xy|| \leq ||x||\,||y||$
- A is complete in the metrix induced by $||.||$

and an involution x^* such that

- $(x+y)^* = x^* + y^*$, $\quad (\lambda x)^* = \bar{\lambda} x^*$
- $(xy)^* = y^* x^*$, $\quad (x^{-1})^* = (x^*)^{-1}$ if x is invertible
- $||xx^*|| = ||x||^2$

Example: commutative C*-algebras

$C(X)$ for a compact Hausdorff space X.

Fact

Self-adjoint, positive properties, $\exp(x)$, $\log(x)$, $x^{\frac{1}{2}}$, etc. can be defined.

C*-algebras

Definition

Associative algebra $\mathcal{A}$ over $\mathbb{C}$ with norm $||x||$ such that

- $||xy|| \leq ||x||\,||y||$
- A is complete in the metrix induced by $||.||$

and an involution x^* such that

- $(x+y)^* = x^* + y^*$, $\quad (\lambda x)^* = \bar{\lambda} x^*$
- $(xy)^* = y^* x^*$, $\quad (x^{-1})^* = (x^*)^{-1}$ if x is invertible
- $||xx^*|| = ||x||^2$

Example: commutative C*-algebras

$C(X)$ for a compact Hausdorff space X.

Fact

Self-adjoint, positive properties, $\exp(x)$, $\log(x)$, $x^{\frac{1}{2}}$, etc. can be defined.

C*-algebras

Definition

Associative algebra $\mathcal{A}$ over $\mathbb{C}$ with norm $||x||$ such that

- $||xy|| \leq ||x|| \, ||y||$
- A is complete in the metrix induced by $||.||$

and an involution x^* such that

- $(x+y)^* = x^* + y^*$, $\qquad (\lambda x)^* = \bar{\lambda} x^*$
- $(xy)^* = y^* x^*$, $\qquad (x^{-1})^* = (x^*)^{-1}$ if x is invertible
- $||xx^*|| = ||x||^2$

Example: commutative C*-algebras

$C(X)$ for a compact Hausdorff space X.

Fact

Self-adjoint, positive properties, $\exp(x)$, $\log(x)$, $x^{\frac{1}{2}}$, etc. can be defined.

Outline

Riemann and Wasserstein metrics

Definition

- The Riemann metric of positive definite matrices A, B is

$$d_R(A,B) = \| \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \|$$

- The Wasserstein metric of p.s.d. matrices A, B is

$$d_W(A,B) = \frac{1}{2} \mathrm{Tr} \left(A + B - 2(A^{\frac{1}{2}} B A^{\frac{1}{2}})^{\frac{1}{2}} \right)$$

- Wasserstein = Leonid Vaseršteĭn
- The Wasserstein metric pops up naturally in the theory of optimal transport
- “earth mover’s distance”
- and also in quantum information theory.

Riemann and Wasserstein metrics

Definition

- The Riemann metric of positive definite matrices A, B is
$$d_R(A, B) = \| \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \|$$
- The Wasserstein metric of p.s.d. matrices A, B is
$$d_W(A, B) = \frac{1}{2} \operatorname{Tr}\left(A + B - 2(A^{\frac{1}{2}} B A^{\frac{1}{2}})^{\frac{1}{2}}\right)$$

- Wasserstein = Leonid Vaseršteĭn
- The Wasserstein metric pops up naturally in the theory of optimal transport
- "earth mover's distance"
- and also in quantum information theory.

Riemann and Wasserstein metrics

Definition

- The Riemann metric of positive definite matrices A, B is

$$d_R(A,B) = \| \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \|$$

- The Wasserstein metric of p.s.d. matrices A, B is

$$d_W(A,B) = \frac{1}{2} \operatorname{Tr}\left(A + B - 2(A^{\frac{1}{2}} B A^{\frac{1}{2}})^{\frac{1}{2}}\right)$$

- Wasserstein = Leonid Vaseršteĭn
- The Wasserstein metric pops up naturally in the theory of optimal transport
- “earth mover’s distance”
- and also in quantum information theory.

Riemann and Wasserstein metrics

Definition

- The Riemann metric of positive definite matrices A, B is

$$d_R(A,B) = \|\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\|$$

- The Wasserstein metric of p.s.d. matrices A, B is

$$d_W(A,B) = \frac{1}{2}\operatorname{Tr}\left(A + B - 2(A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{1}{2}}\right)$$

- Wasserstein = Leonid Vaseršteĭn
- The Wasserstein metric pops up naturally in the theory of optimal transport
- “earth mover’s distance”
- and also in quantum information theory.

Riemann and Wasserstein metrics

Definition

- The Riemann metric of positive definite matrices A, B is

$$d_R(A,B) = \| \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \|$$

- The Wasserstein metric of p.s.d. matrices A, B is

$$d_W(A,B) = \frac{1}{2} \operatorname{Tr}\left(A + B - 2(A^{\frac{1}{2}} B A^{\frac{1}{2}})^{\frac{1}{2}} \right)$$

- Wasserstein = Leonid Vaseršteĭn
- The Wasserstein metric pops up naturally in the theory of optimal transport
- “earth mover’s distance”
- and also in quantum information theory.

Riemann and Wasserstein metrics

Definition

- The Riemann metric of positive definite matrices A, B is

$$d_R(A, B) = \| \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \|$$

- The Wasserstein metric of p.s.d. matrices A, B is

$$d_W(A, B) = \frac{1}{2} \operatorname{Tr} \left(A + B - 2(A^{\frac{1}{2}} B A^{\frac{1}{2}})^{\frac{1}{2}} \right)$$

- Wasserstein = Leonid Vaseršteĭn
- The Wasserstein metric pops up naturally in the theory of optimal transport
- “earth mover’s distance”
- and also in quantum information theory.

Riemann and Wasserstein metrics

Definition

- The Riemann metric of positive definite matrices A, B is
$$d_R(A, B) = \|\log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})\|$$
- The Wasserstein metric of p.s.d. matrices A, B is
$$d_W(A, B) = \frac{1}{2} \operatorname{Tr}\left(A + B - 2(A^{\frac{1}{2}} B A^{\frac{1}{2}})^{\frac{1}{2}}\right)$$

- Wasserstein = Leonid Vaseršteĭn
- The Wasserstein metric pops up naturally in the theory of optimal transport
- “earth mover’s distance”
- and also in quantum information theory.

Least square means and midpoints

Definition

Let (X, d) be a metric space, and $p_1, \ldots, p_m$ points in X.

- The least square mean is the point $x \in X$ which minimizes
$$d(x, p_1)^2 + \cdots + d(x, p_m)^2.$$
- The weighted mean with weights $0 \leq w_1, \ldots, w_m$ minimizes
$$w_1 d(x, p_1)^2 + \cdots + w_1 d(x, p_m)^2.$$
- The midpoint of two points p, q is the least square mean of p, q.
- The segment pq consist of the weighted means of p, q with weights $1 - t, t$, $0 \leq t \leq 1$.

Example

In the Euclidean space, the (weighted) least square mean is the (weighted) arithmetic mean.

Least square means and midpoints

Definition

Let (X, d) be a metric space, and $p_1, \ldots, p_m$ points in X.

- The least square mean is the point $x \in X$ which minimizes
$$d(x, p_1)^2 + \cdots + d(x, p_m)^2.$$
- The weighted mean with weights $0 \leq w_1, \ldots, w_m$ minimizes
$$w_1 d(x, p_1)^2 + \cdots + w_1 d(x, p_m)^2.$$
- The midpoint of two points p, q is the least square mean of p, q.
- The segment pq consist of the weighted means of p, q with weights $1 - t, t$, $0 \leq t \leq 1$.

Example

In the Euclidean space, the (weighted) least square mean is the (weighted) arithmetic mean.

Least square means and midpoints

Definition

Let (X, d) be a metric space, and $p_1, \ldots, p_m$ points in X.

- The least square mean is the point $x \in X$ which minimizes
$$d(x, p_1)^2 + \cdots + d(x, p_m)^2.$$
- The weighted mean with weights $0 \le w_1, \ldots, w_m$ minimizes
$$w_1 d(x, p_1)^2 + \cdots + w_1 d(x, p_m)^2.$$
- The midpoint of two points p, q is the least square mean of p, q.
- The segment pq consist of the weighted means of p, q with weights $1 - t, t$, $0 \le t \le 1$.

Example

In the Euclidean space, the (weighted) least square mean is the (weighted) arithmetic mean.

Least square means and midpoints

Definition

Let (X, d) be a metric space, and $p_1, \ldots, p_m$ points in X.

- The least square mean is the point $x \in X$ which minimizes

$$d(x, p_1)^2 + \cdots + d(x, p_m)^2.$$

- The weighted mean with weights $0 \leq w_1, \ldots, w_m$ minimizes

$$w_1 d(x, p_1)^2 + \cdots + w_1 d(x, p_m)^2.$$

- The midpoint of two points p, q is the least square mean of p, q.
- The segment pq consist of the weighted means of p, q with weights $1 - t, t$, $0 \leq t \leq 1$.

Example

In the Euclidean space, the (weighted) least square mean is the (weighted) arithmetic mean.

Least square means and midpoints

Definition

Let (X, d) be a metric space, and $p_1, \ldots, p_m$ points in X.

- The least square mean is the point $x \in X$ which minimizes
$$d(x, p_1)^2 + \cdots + d(x, p_m)^2.$$
- The weighted mean with weights $0 \leq w_1, \ldots, w_m$ minimizes
$$w_1 d(x, p_1)^2 + \cdots + w_1 d(x, p_m)^2.$$
- The midpoint of two points p, q is the least square mean of p, q.
- The segment pq consist of the weighted means of p, q with weights $1 - t, t$, $0 \leq t \leq 1$.

Example

In the Euclidean space, the (weighted) least square mean is the (weighted) arithmetic mean.

Least square means and midpoints

Definition

Let (X, d) be a metric space, and $p_1, \ldots, p_m$ points in X.

- The least square mean is the point $x \in X$ which minimizes
$$d(x, p_1)^2 + \cdots + d(x, p_m)^2.$$
- The weighted mean with weights $0 \leq w_1, \ldots, w_m$ minimizes
$$w_1 d(x, p_1)^2 + \cdots + w_1 d(x, p_m)^2.$$
- The midpoint of two points p, q is the least square mean of p, q.
- The segment pq consist of the weighted means of p, q with weights $1 - t, t$, $0 \leq t \leq 1$.

Example

In the Euclidean space, the (weighted) least square mean is the (weighted) arithmetic mean.

The Riemannian geometric mean

midpoint: $x \# y = x^{\frac{1}{2}} \left(x^{-\frac{1}{2}}\, y\, x^{-\frac{1}{2}} \right)^{\frac{1}{2}} x^{\frac{1}{2}}$

weighted midpoint: $x \#_t y = x^{\frac{1}{2}} \left(x^{-\frac{1}{2}}\, y\, x^{-\frac{1}{2}} \right)^{t} x^{\frac{1}{2}}$

- **Central reflection** of x on z: solve $x \# y = z$ for y:

$$x^{\frac{1}{2}} \left(x^{-\frac{1}{2}}\, y\, x^{-\frac{1}{2}} \right)^{\frac{1}{2}} x^{\frac{1}{2}} = z$$

$$x^{-\frac{1}{2}}\, y\, x^{-\frac{1}{2}} = \left(x^{-\frac{1}{2}}\, z\, x^{-\frac{1}{2}} \right)^{2} = x^{-\frac{1}{2}}\, z x^{-1} z\, x^{-\frac{1}{2}}$$

$$y = z x^{-1} z$$

- No square root is needed!
- **Translation =** product of two central reflections: $x^{\frac{1}{2}} y x^{\frac{1}{2}}$.

The Riemannian geometric mean

midpoint: $$x \# y = x^{\frac{1}{2}} \left(x^{-\frac{1}{2}}\, y\, x^{-\frac{1}{2}} \right)^{\frac{1}{2}} x^{\frac{1}{2}}$$

weighted midpoint: $$x \#_t y = x^{\frac{1}{2}} \left(x^{-\frac{1}{2}}\, y\, x^{-\frac{1}{2}} \right)^{t} x^{\frac{1}{2}}$$

- **Central reflection** of x on z: solve $x \# y = z$ for y:

$$x^{\frac{1}{2}} \left(x^{-\frac{1}{2}}\, y\, x^{-\frac{1}{2}} \right)^{\frac{1}{2}} x^{\frac{1}{2}} = z$$

$$x^{-\frac{1}{2}}\, y\, x^{-\frac{1}{2}} = \left(x^{-\frac{1}{2}}\, z\, x^{-\frac{1}{2}} \right)^{2} = x^{-\frac{1}{2}}\, z x^{-1} z\, x^{-\frac{1}{2}}$$

$$y = z x^{-1} z$$

- No square root is needed!
- **Translation =** product of two central reflections: $x^{\frac{1}{2}} y x^{\frac{1}{2}}$.

The Riemannian geometric mean

midpoint: $$x \# y = x^{\frac{1}{2}} \left(x^{-\frac{1}{2}} \, y \, x^{-\frac{1}{2}} \right)^{\frac{1}{2}} x^{\frac{1}{2}}$$

weighted midpoint: $$x \#_t y = x^{\frac{1}{2}} \left(x^{-\frac{1}{2}} \, y \, x^{-\frac{1}{2}} \right)^{t} x^{\frac{1}{2}}$$

- **Central reflection** of x on z: solve $x \# y = z$ for y:

$$x^{\frac{1}{2}} \left(x^{-\frac{1}{2}} \, y \, x^{-\frac{1}{2}} \right)^{\frac{1}{2}} x^{\frac{1}{2}} = z$$

$$x^{-\frac{1}{2}} \, y \, x^{-\frac{1}{2}} = \left(x^{-\frac{1}{2}} \, z \, x^{-\frac{1}{2}} \right)^{2} = x^{-\frac{1}{2}} \, z x^{-1} z \, x^{-\frac{1}{2}}$$

$$y = z x^{-1} z$$

- No square root is needed!
- **Translation =** product of two central reflections: $x^{\frac{1}{2}} y x^{\frac{1}{2}}$.

The Riemannian geometric mean

midpoint: $$x\#y = x^{\frac{1}{2}}\left(x^{-\frac{1}{2}}\,y\,x^{-\frac{1}{2}}\right)^{\frac{1}{2}}x^{\frac{1}{2}}$$

weighted midpoint: $$x\#_t y = x^{\frac{1}{2}}\left(x^{-\frac{1}{2}}\,y\,x^{-\frac{1}{2}}\right)^{t}x^{\frac{1}{2}}$$

- **Central reflection** of x on z: solve $x\#y = z$ for y:

$$x^{\frac{1}{2}}\left(x^{-\frac{1}{2}}\,y\,x^{-\frac{1}{2}}\right)^{\frac{1}{2}}x^{\frac{1}{2}} = z$$

$$x^{-\frac{1}{2}}\,y\,x^{-\frac{1}{2}} = \left(x^{-\frac{1}{2}}\,z\,x^{-\frac{1}{2}}\right)^{2} = x^{-\frac{1}{2}}\,zx^{-1}z\,x^{-\frac{1}{2}}$$

$$y = zx^{-1}z$$

- No square root is needed!
- **Translation =** product of two central reflections: $x^{\frac{1}{2}}yx^{\frac{1}{2}}$.

The Wasserstein mean

$$x \diamond y = \frac{1}{4} x^{-\frac{1}{2}} \left(x + \sqrt{x^{\frac{1}{2}} y x^{\frac{1}{2}}} \right)^2 x^{-\frac{1}{2}}$$

- Straightforward computations [Bhatia, Jain, Lim 2019] give that

$$x \diamond y = \frac{1}{4} \left(x + y + x(x^{-1} \# y) + (x^{-1} \# y) x \right).$$

- and

$$(x(x^{-1} \# y))^2 = xy, \quad ((x^{-1} \# y) x)^2 = yx.$$

- With some cheating, we have

$$x \diamond y = \frac{1}{4} \left(x + y + (xy)^{\frac{1}{2}} + (yx)^{\frac{1}{2}} \right).$$

- x, y commute if and only if

$$x \diamond y = \left(\frac{\sqrt{x} + \sqrt{y}}{2} \right)^2.$$

The Wasserstein mean

$$x \diamond y = \frac{1}{4} x^{-\frac{1}{2}} \left(x + \sqrt{x^{\frac{1}{2}} y x^{\frac{1}{2}}} \right)^2 x^{-\frac{1}{2}}$$

- Straightforward computations [Bhatia, Jain, Lim 2019] give that

$$x \diamond y = \frac{1}{4} \big(x + y + x(x^{-1} \# y) + (x^{-1} \# y) x \big).$$

- and

$$(x(x^{-1} \# y))^2 = xy, \quad ((x^{-1} \# y)x)^2 = yx.$$

- With some cheating, we have

$$x \diamond y = \frac{1}{4} \left(x + y + (xy)^{\frac{1}{2}} + (yx)^{\frac{1}{2}} \right).$$

- x, y commute if and only if

$$x \diamond y = \left(\frac{\sqrt{x} + \sqrt{y}}{2} \right)^2.$$

The Wasserstein mean

$$x \diamond y = \frac{1}{4} x^{-\frac{1}{2}} \left(x + \sqrt{x^{\frac{1}{2}} y x^{\frac{1}{2}}} \right)^2 x^{-\frac{1}{2}}$$

- Straightforward computations [Bhatia, Jain, Lim 2019] give that

$$x \diamond y = \frac{1}{4} \left(x + y + x(x^{-1} \# y) + (x^{-1} \# y) x \right).$$

- and

$$(x(x^{-1} \# y))^2 = xy, \quad ((x^{-1} \# y) x)^2 = yx.$$

- With some cheating, we have

$$x \diamond y = \frac{1}{4} \left(x + y + (xy)^{\frac{1}{2}} + (yx)^{\frac{1}{2}} \right).$$

- x, y commute if and only if

$$x \diamond y = \left(\frac{\sqrt{x} + \sqrt{y}}{2} \right)^2.$$

The Wasserstein mean

$$x \diamond y = \frac{1}{4} x^{-\frac{1}{2}} \left(x + \sqrt{x^{\frac{1}{2}} y x^{\frac{1}{2}}} \right)^2 x^{-\frac{1}{2}}$$

- Straightforward computations [Bhatia, Jain, Lim 2019] give that

$$x \diamond y = \frac{1}{4} \left(x + y + x(x^{-1} \# y) + (x^{-1} \# y) x \right).$$

- and

$$(x(x^{-1} \# y))^2 = xy, \quad ((x^{-1} \# y) x)^2 = yx.$$

- With some cheating, we have

$$x \diamond y = \frac{1}{4} \left(x + y + (xy)^{\frac{1}{2}} + (yx)^{\frac{1}{2}} \right).$$

- x, y commute if and only if

$$x \diamond y = \left(\frac{\sqrt{x} + \sqrt{y}}{2} \right)^2.$$

The Wasserstein mean

$$x \diamond y = \frac{1}{4} x^{-\frac{1}{2}} \left(x + \sqrt{x^{\frac{1}{2}} y x^{\frac{1}{2}}} \right)^2 x^{-\frac{1}{2}}$$

- Straightforward computations [Bhatia, Jain, Lim 2019] give that

$$x \diamond y = \frac{1}{4} \left(x + y + x(x^{-1} \# y) + (x^{-1} \# y) x \right).$$

- and

$$(x(x^{-1} \# y))^2 = xy, \quad ((x^{-1} \# y) x)^2 = yx.$$

- With some cheating, we have

$$x \diamond y = \frac{1}{4} \left(x + y + (xy)^{\frac{1}{2}} + (yx)^{\frac{1}{2}} \right).$$

- x, y commute if and only if

$$x \diamond y = \left(\frac{\sqrt{x} + \sqrt{y}}{2} \right)^2.$$

Outline

Molnár's results

Using heavy C*-algebra machinery, Lajos Molnár (Szeged) proved:

Theorem (Molnár 2023)

Let $\mathcal{A}$ be a C*-algebra and $\mathcal{A}^{++}$ the set of positive definite elements. The bijective map $\phi : \mathcal{A}^{++} \to \mathcal{A}^{++}$ preserves the Wasserstein mean if and only if there is

- a Jordan *-automorphism $J : \mathcal{A} \to \mathcal{A}$
- and a central element $c \in \mathcal{A}$

such that

$$\phi(x) = cJ(x), \qquad x \in \mathcal{A}^{++}.$$

Theorem (Molnár 2023)

The binary operation $4(x \diamond y)$ is left alternative, right alternative and flexible. Moreover, it is a semigroup (i.e. associative) if and only if the algebra $\mathcal{A}$ is commutative.

Molnár's results

Using heavy C*-algebra machinery, Lajos Molnár (Szeged) proved:

Theorem (Molnár 2023)

Let $\mathcal{A}$ be a C*-algebra and $\mathcal{A}^{++}$ the set of positive definite elements. The bijective map $\phi : \mathcal{A}^{++} \to \mathcal{A}^{++}$ preserves the Wasserstein mean if and only if there is

- a Jordan *-automorphism $J : \mathcal{A} \to \mathcal{A}$
- and a central element $c \in \mathcal{A}$

such that

$$\phi(x) = cJ(x), \qquad x \in \mathcal{A}^{++}.$$

Theorem (Molnár 2023)

The binary operation $4(x \diamond y)$ is left alternative, right alternative and flexible. Moreover, it is a semigroup (i.e. associative) if and only if the algebra $\mathcal{A}$ is commutative.

Molnár's results

Using heavy C*-algebra machinery, Lajos Molnár (Szeged) proved:

Theorem (Molnár 2023)

Let $\mathcal{A}$ be a C*-algebra and $\mathcal{A}^{++}$ the set of positive definite elements. The bijective map $\phi : \mathcal{A}^{++} \to \mathcal{A}^{++}$ preserves the Wasserstein mean if and only if there is

- a Jordan *-automorphism $J : \mathcal{A} \to \mathcal{A}$
- and a central element $c \in \mathcal{A}$

such that

$$\phi(x) = cJ(x), \qquad x \in \mathcal{A}^{++}.$$

Theorem (Molnár 2023)

The binary operation $4(x \diamond y)$ is left alternative, right alternative and flexible. Moreover, it is a semigroup (i.e. associative) if and only if the algebra $\mathcal{A}$ is commutative.

Molnár's results

Using heavy C*-algebra machinery, Lajos Molnár (Szeged) proved:

Theorem (Molnár 2023)

Let $\mathcal{A}$ be a C*-algebra and $\mathcal{A}^{++}$ the set of positive definite elements. The bijective map $\phi : \mathcal{A}^{++} \to \mathcal{A}^{++}$ preserves the Wasserstein mean if and only if there is

- a Jordan *-automorphism $J : \mathcal{A} \to \mathcal{A}$
- and a central element $c \in \mathcal{A}$

such that

$$\phi(x) = cJ(x), \qquad x \in \mathcal{A}^{++}.$$

Theorem (Molnár 2023)

The binary operation $4(x \diamond y)$ is left alternative, right alternative and flexible. Moreover, it is a semigroup (i.e. associative) if and only if the algebra $\mathcal{A}$ is commutative.

The algebraization project

- Define algebraic structures in which the Riemannian geometric mean and the Wasserstein mean can be defined.

$$x \# y = x^{\frac{1}{2}} \left(x^{-\frac{1}{2}} \, y \, x^{-\frac{1}{2}} \right)^{\frac{1}{2}} x^{\frac{1}{2}}$$

$$x \diamond y = \frac{1}{4} x^{-\frac{1}{2}} \left(x + \sqrt{x^{\frac{1}{2}} y x^{\frac{1}{2}}} \right)^2 x^{-\frac{1}{2}}$$

- Prove Molnár-type results for these structures.

The algebraization project

- Define algebraic structures in which the Riemannian geometric mean and the Wasserstein mean can be defined.

$$x \# y = x^{\frac{1}{2}} \left(x^{-\frac{1}{2}} \, y \, x^{-\frac{1}{2}} \right)^{\frac{1}{2}} x^{\frac{1}{2}}$$

$$x \diamond y = \frac{1}{4} x^{-\frac{1}{2}} \left(x + \sqrt{x^{\frac{1}{2}} y x^{\frac{1}{2}}} \right)^{2} x^{-\frac{1}{2}}$$

- Prove Molnár-type results for these structures.

The success story: Twisted subgroups and Bol loops

Definition (Aschbacher)

Let G be a group. The subset $S \subseteq G$ is a twisted subgroup of G is

- $1 \in S$,
- $a^{-1} \in S$ for all $a \in S$,
- $aba \in S$ for all $a, b \in S$.

Examples

Let α be an involutorial automorphism of G.

- The set $S = \{x \in G \mid \alpha(x) = x^{-1}\}$ of anti-fixed elements.
- The set $T = \{x^{-1}\alpha(x) \mid x \in G\}$ of "commutators".

- Homework: Construct the set of positive definite matrices as set of commutators.

The success story: Twisted subgroups and Bol loops

Definition (Aschbacher)

Let G be a group. The subset $S \subseteq G$ is a twisted subgroup of G is

- $1 \in S$,
- $a^{-1} \in S$ for all $a \in S$,
- $aba \in S$ for all $a, b \in S$.

Examples

Let α be an involutorial automorphism of G.

- The set $S = \{x \in G \mid \alpha(x) = x^{-1}\}$ of anti-fixed elements.
- The set $T = \{x^{-1}\alpha(x) \mid x \in G\}$ of "commutators".

- Homework: Construct the set of positive definite matrices as set of commutators.

The success story: Twisted subgroups and Bol loops

Definition (Aschbacher)

Let G be a group. The subset $S \subseteq G$ is a twisted subgroup of G is

- $1 \in S$,
- $a^{-1} \in S$ for all $a \in S$,
- $aba \in S$ for all $a, b \in S$.

Examples

Let α be an involutorial automorphism of G.

- The set $S = \{x \in G \mid \alpha(x) = x^{-1}\}$ of anti-fixed elements.
- The set $T = \{x^{-1}\alpha(x) \mid x \in G\}$ of "commutators".

- Homework: Construct the set of positive definite matrices as set of commutators.

The success story: Twisted subgroups and Bol loops

Definition (Aschbacher)

Let G be a group. The subset $S \subseteq G$ is a twisted subgroup of G is

- $1 \in S$,
- $a^{-1} \in S$ for all $a \in S$,
- $aba \in S$ for all $a, b \in S$.

Examples

Let α be an involutorial automorphism of G.

- The set $S = \{x \in G \mid \alpha(x) = x^{-1}\}$ of anti-fixed elements.
- The set $T = \{x^{-1}\alpha(x) \mid x \in G\}$ of "commutators".

- Homework: Construct the set of positive definite matrices as set of commutators.

The success story: Twisted subgroups and Bol loops

Definition (Aschbacher)

Let G be a group. The subset $S \subseteq G$ is a twisted subgroup of G is

- $1 \in S$,
- $a^{-1} \in S$ for all $a \in S$,
- $aba \in S$ for all $a, b \in S$.

Examples

Let α be an involutorial automorphism of G.

- The set $S = \{x \in G \mid \alpha(x) = x^{-1}\}$ of anti-fixed elements.
- The set $T = \{x^{-1}\alpha(x) \mid x \in G\}$ of "commutators".

- Homework: Construct the set of positive definite matrices as set of commutators.

The success story: Twisted subgroups and Bol loops

Definition (Aschbacher)

Let G be a group. The subset $S \subseteq G$ is a twisted subgroup of G is

- $1 \in S$,
- $a^{-1} \in S$ for all $a \in S$,
- $aba \in S$ for all $a, b \in S$.

Examples

Let α be an involutorial automorphism of G.

- The set $S = \{x \in G \mid \alpha(x) = x^{-1}\}$ of anti-fixed elements.
- The set $T = \{x^{-1}\alpha(x) \mid x \in G\}$ of "commutators".

- Homework: Construct the set of positive definite matrices as set of commutators.

The success story: Twisted subgroups and Bol loops

Definition (Aschbacher)

Let G be a group. The subset $S \subseteq G$ is a twisted subgroup of G is

- $1 \in S$,
- $a^{-1} \in S$ for all $a \in S$,
- $aba \in S$ for all $a, b \in S$.

Examples

Let α be an involutorial automorphism of G.

- The set $S = \{x \in G \mid \alpha(x) = x^{-1}\}$ of anti-fixed elements.
- The set $T = \{x^{-1}\alpha(x) \mid x \in G\}$ of "commutators".

- Homework: Construct the set of positive definite matrices as set of commutators.

The success story: Twisted subgroups and Bol loops

Definition (Aschbacher)

Let G be a group. The subset $S \subseteq G$ is a twisted subgroup of G is

- $1 \in S$,
- $a^{-1} \in S$ for all $a \in S$,
- $aba \in S$ for all $a, b \in S$.

Examples

Let α be an involutorial automorphism of G.

- The set $S = \{x \in G \mid \alpha(x) = x^{-1}\}$ of anti-fixed elements.
- The set $T = \{x^{-1}\alpha(x) \mid x \in G\}$ of “commutators”.

- Homework: Construct the set of positive definite matrices as set of commutators.

Bol loops, Bruck loops and the Z*-theorem

Theorem [Glauberman 1968 (and Kiechle, and Ungar, and ...)]

- The reflection
$$xy^{-1}x$$
is well-defined in any twisted subgroup S.
- It is a left quasigroup.
- Moreover, it is a quasigroup if and only if S is 2-divisible.
- In the latter case, its loop isotopy is given by the operation $x^{\frac{1}{2}}yx^{\frac{1}{2}}$.

(A version of) Glauberman's Z*-theorem

If S is a finite twisted subgroup of odd order, then $\langle S\rangle$ has odd order. In particular, the loop $x^{\frac{1}{2}}yx^{\frac{1}{2}}$ is solvable.

Interesting objects (Aschbacher 2006, GN 2007, Baumeister, Stein, Stroth ~2010)

Finite twisted subgroups (Bol loops) of exponent two.

Bol loops, Bruck loops and the Z*-theorem

Theorem [Glauberman 1968 (and Kiechle, and Ungar, and ...)]

- The reflection
  $$xy^{-1}x$$
  is well-defined in any twisted subgroup S.
- It is a left quasigroup.
- Moreover, it is a quasigroup if and only if S is 2-divisible.
- In the latter case, its loop isotopy is given by the operation $x^{\frac{1}{2}}yx^{\frac{1}{2}}$.

(A version of) Glauberman's Z*-theorem

If S is a finite twisted subgroup of odd order, then $\langle S\rangle$ has odd order. In particular, the loop $x^{\frac{1}{2}}yx^{\frac{1}{2}}$ is solvable.

Interesting objects (Aschbacher 2006, GN 2007, Baumeister, Stein, Stroth ∼2010)

Finite twisted subgroups (Bol loops) of exponent two.

Bol loops, Bruck loops and the Z*-theorem

Theorem [Glauberman 1968 (and Kiechle, and Ungar, and ...)]

- The reflection
$$xy^{-1}x$$
is well-defined in any twisted subgroup S.
- It is a left quasigroup.
- Moreover, it is a quasigroup if and only if S is 2-divisible.
- In the latter case, its loop isotopy is given by the operation $x^{\frac{1}{2}}yx^{\frac{1}{2}}$.

(A version of) Glauberman's Z*-theorem

If S is a finite twisted subgroup of odd order, then $\langle S\rangle$ has odd order. In particular, the loop $x^{\frac{1}{2}}yx^{\frac{1}{2}}$ is solvable.

Interesting objects (Aschbacher 2006, GN 2007, Baumeister, Stein, Stroth ~2010)

Finite twisted subgroups (Bol loops) of exponent two.

Bol loops, Bruck loops and the Z*-theorem

Theorem [Glauberman 1968 (and Kiechle, and Ungar, and ...)]

- The reflection
$$xy^{-1}x$$
is well-defined in any twisted subgroup S.
- It is a left quasigroup.
- Moreover, it is a quasigroup if and only if S is 2-divisible.
- In the latter case, its loop isotopy is given by the operation $x^{\frac{1}{2}}yx^{\frac{1}{2}}$.

(A version of) Glauberman's Z*-theorem

If S is a finite twisted subgroup of odd order, then $\langle S\rangle$ has odd order. In particular, the loop $x^{\frac{1}{2}}yx^{\frac{1}{2}}$ is solvable.

Interesting objects (Aschbacher 2006, GN 2007, Baumeister, Stein, Stroth ~2010)

Finite twisted subgroups (Bol loops) of exponent two.

Bol loops, Bruck loops and the Z*-theorem

Theorem [Glauberman 1968 (and Kiechle, and Ungar, and ...)]

- The reflection
$$xy^{-1}x$$
is well-defined in any twisted subgroup S.
- It is a left quasigroup.
- Moreover, it is a quasigroup if and only if S is 2-divisible.
- In the latter case, its loop isotopy is given by the operation $x^{\frac{1}{2}}yx^{\frac{1}{2}}$.

(A version of) Glauberman's Z*-theorem

If S is a finite twisted subgroup of odd order, then $\langle S \rangle$ has odd order. In particular, the loop $x^{\frac{1}{2}}yx^{\frac{1}{2}}$ is solvable.

Interesting objects (Aschbacher 2006, GN 2007, Baumeister, Stein, Stroth ~2010)

Finite twisted subgroups (Bol loops) of exponent two.

Bol loops, Bruck loops and the Z*-theorem

Theorem [Glauberman 1968 (and Kiechle, and Ungar, and ...)]

- The reflection
$$xy^{-1}x$$
is well-defined in any twisted subgroup S.
- It is a left quasigroup.
- Moreover, it is a quasigroup if and only if S is 2-divisible.
- In the latter case, its loop isotopy is given by the operation $x^{\frac{1}{2}}yx^{\frac{1}{2}}$.

(A version of) Glauberman's Z*-theorem

If S is a finite twisted subgroup of odd order, then $\langle S\rangle$ has odd order. In particular, the loop $x^{\frac{1}{2}}yx^{\frac{1}{2}}$ is solvable.

Interesting objects (Aschbacher 2006, GN 2007, Baumeister, Stein, Stroth ~2010)

Finite twisted subgroups (Bol loops) of exponent two.

Bol loops, Bruck loops and the Z*-theorem

Theorem [Glauberman 1968 (and Kiechle, and Ungar, and ...)]

- The reflection
$$xy^{-1}x$$
is well-defined in any twisted subgroup S.
- It is a left quasigroup.
- Moreover, it is a quasigroup if and only if S is 2-divisible.
- In the latter case, its loop isotopy is given by the operation $x^{\frac{1}{2}}yx^{\frac{1}{2}}$.

(A version of) Glauberman's Z*-theorem

If S is a finite twisted subgroup of odd order, then $\langle S \rangle$ has odd order. In particular, the loop $x^{\frac{1}{2}}yx^{\frac{1}{2}}$ is solvable.

Interesting objects (Aschbacher 2006, GN 2007, Baumeister, Stein, Stroth ∼2010)

Finite twisted subgroups (Bol loops) of exponent two.

Unipotent matrices

(Upper) unipotent matrix

$$\begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Observations

- Convex combination of unipotent matrices are unipotent.
- $A^n = 1$ for $n \times n$ unipotent A.
- Inverse and square root are well-defined for unipotent matrices.
- So is the Wasserstein mean

$$x \diamond y = \frac{1}{4} x^{-\frac{1}{2}} \left(x + \sqrt{x^{\frac{1}{2}} y x^{\frac{1}{2}}} \right)^2 x^{-\frac{1}{2}}.$$

- $4(x \diamond y)$ is not!

Unipotent matrices

(Upper) unipotent matrix

$$\begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Observations

- Convex combination of unipotent matrices are unipotent.
- $A^n = 1$ for $n \times n$ unipotent A.
- Inverse and square root are well-defined for unipotent matrices.
- So is the Wasserstein mean

$$x \diamond y = \frac{1}{4} x^{-\frac{1}{2}} \left(x + \sqrt{x^{\frac{1}{2}} y x^{\frac{1}{2}}} \right)^2 x^{-\frac{1}{2}}.$$

- $4(x \diamond y)$ is not!

Unipotent matrices

(Upper) unipotent matrix

$$\begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Observations

- Convex combination of unipotent matrices are unipotent.
- $A^n = 1$ for $n \times n$ unipotent A.
- Inverse and square root are well-defined for unipotent matrices.
- So is the Wasserstein mean

$$x \diamond y = \frac{1}{4} x^{-\frac{1}{2}} \left(x + \sqrt{x^{\frac{1}{2}} y x^{\frac{1}{2}}} \right)^2 x^{-\frac{1}{2}}.$$

- $4(x \diamond y)$ is not!

Unipotent matrices

(Upper) unipotent matrix

$$\begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Observations

- Convex combination of unipotent matrices are unipotent.
- $A^n = 1$ for $n \times n$ unipotent A.
- Inverse and square root are well-defined for unipotent matrices.
- So is the Wasserstein mean

$$x \diamond y = \frac{1}{4} x^{-\frac{1}{2}} \left(x + \sqrt{x^{\frac{1}{2}} y x^{\frac{1}{2}}} \right)^2 x^{-\frac{1}{2}}.$$

- $4(x \diamond y)$ is not!

Unipotent matrices

(Upper) unipotent matrix

$$\begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Observations

- Convex combination of unipotent matrices are unipotent.
- $A^n = 1$ for $n \times n$ unipotent A.
- Inverse and square root are well-defined for unipotent matrices.
- So is the Wasserstein mean

$$x \diamond y = \frac{1}{4} x^{-\frac{1}{2}} \left(x + \sqrt{x^{\frac{1}{2}} y x^{\frac{1}{2}}} \right)^2 x^{-\frac{1}{2}}.$$

- $4(x \diamond y)$ is not!

Unipotent matrices

(Upper) unipotent matrix

$$\begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Observations

- Convex combination of unipotent matrices are unipotent.
- $A^n = 1$ for $n \times n$ unipotent A.
- Inverse and square root are well-defined for unipotent matrices.
- So is the Wasserstein mean

$$x \diamond y = \frac{1}{4} x^{-\frac{1}{2}} \left(x + \sqrt{x^{\frac{1}{2}} y x^{\frac{1}{2}}} \right)^2 x^{-\frac{1}{2}}.$$

- $4(x \diamond y)$ is not!

Unipotent matrices

(Upper) unipotent matrix

$$\begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Observations

- Convex combination of unipotent matrices are unipotent.
- $A^n = 1$ for $n \times n$ unipotent A.
- Inverse and square root are well-defined for unipotent matrices.
- So is the Wasserstein mean

$$x \diamond y = \frac{1}{4} x^{-\frac{1}{2}} \left(x + \sqrt{x^{\frac{1}{2}} y x^{\frac{1}{2}}} \right)^2 x^{-\frac{1}{2}}.$$

- $4(x \diamond y)$ is not!

Split C*-by-nilpotent algebras

Definition: Split C*-by-nilpotent algebra

Let B be a unitary associative algebra over $\mathbb{C}$. We say that B is a split C^*-by-nilpotent algebra, if the following conditions hold:

1. B has a unitary subalgebra $\mathcal{A}$ that is a C^*-algebra and the unit of $\mathcal{A}$ coincides with the unit of B;
2. B has an ideal N which consists of nilpotent elements;
3. $B = \mathcal{A} + N$;
4. The elements of $\mathcal{A}$ and N commute.

Lemma

The geometric and Wasserstein means, and $4(x \diamond y)$ is well-defined on

$$B^{++} = \{a + n \mid a \in \mathcal{A}^{++}, n \in N\}.$$

Split C*-by-nilpotent algebras

Definition: Split C*-by-nilpotent algebra

Let B be a unitary associative algebra over $\mathbb{C}$. We say that B is a split C^*-by-nilpotent algebra, if the following conditions hold:

1. B has a unitary subalgebra $\mathcal{A}$ that is a C^*-algebra and the unit of $\mathcal{A}$ coincides with the unit of B;
2. B has an ideal N which consists of nilpotent elements;
3. $B = \mathcal{A} + N$;
4. The elements of $\mathcal{A}$ and N commute.

Lemma

The geometric and Wasserstein means, and $4(x \diamond y)$ is well-defined on

$$B^{++} = \{a + n \mid a \in \mathcal{A}^{++}, n \in N\}.$$

Split C*-by-nilpotent algebras

Definition: Split C*-by-nilpotent algebra

Let B be a unitary associative algebra over $\mathbb{C}$. We say that B is a split C^*-by-nilpotent algebra, if the following conditions hold:

1. B has a unitary subalgebra $\mathcal{A}$ that is a C^*-algebra and the unit of $\mathcal{A}$ coincides with the unit of B;
2. B has an ideal N which consists of nilpotent elements;
3. $B = \mathcal{A} + N$;
4. The elements of $\mathcal{A}$ and N commute.

Lemma

The geometric and Wasserstein means, and $4(x \diamond y)$ is well-defined on

$$B^{++} = \{a + n \mid a \in \mathcal{A}^{++}, n \in N\}.$$

Split C*-by-nilpotent algebras

Definition: Split C*-by-nilpotent algebra

Let B be a unitary associative algebra over $\mathbb{C}$. We say that B is a split C^*-by-nilpotent algebra, if the following conditions hold:

1. B has a unitary subalgebra $\mathcal{A}$ that is a C^*-algebra and the unit of $\mathcal{A}$ coincides with the unit of B;
2. B has an ideal N which consists of nilpotent elements;
3. $B = \mathcal{A} + N$;
4. The elements of $\mathcal{A}$ and N commute.

Lemma

The geometric and Wasserstein means, and $4(x \diamond y)$ is well-defined on

$$B^{++} = \{a + n \mid a \in \mathcal{A}^{++}, n \in N\}.$$

Split C*-by-nilpotent algebras

Definition: Split C*-by-nilpotent algebra

Let B be a unitary associative algebra over $\mathbb{C}$. We say that B is a split C^*-by-nilpotent algebra, if the following conditions hold:

1. B has a unitary subalgebra $\mathcal{A}$ that is a C^*-algebra and the unit of $\mathcal{A}$ coincides with the unit of B;
2. B has an ideal N which consists of nilpotent elements;
3. $B = \mathcal{A} + N$;
4. The elements of $\mathcal{A}$ and N commute.

Lemma

The geometric and Wasserstein means, and $4(x \diamond y)$ is well-defined on

$$B^{++} = \{a + n \mid a \in \mathcal{A}^{++}, n \in N\}.$$

Split C*-by-nilpotent algebras

Definition: Split C*-by-nilpotent algebra

Let B be a unitary associative algebra over $\mathbb{C}$. We say that B is a split C^*-by-nilpotent algebra, if the following conditions hold:

1. B has a unitary subalgebra $\mathcal{A}$ that is a C^*-algebra and the unit of $\mathcal{A}$ coincides with the unit of B;
2. B has an ideal N which consists of nilpotent elements;
3. $B = \mathcal{A} + N$;
4. The elements of $\mathcal{A}$ and N commute.

Lemma

The geometric and Wasserstein means, and $4(x \diamond y)$ is well-defined on

$$B^{++} = \{a + n \mid a \in \mathcal{A}^{++}, n \in N\}.$$

Associativity of $4(x \diamond y)$

Theorem (La Rosa, Mancini, GN 2024)

Let $B = \mathcal{A} \oplus N$ be a split C^*-by-nilpotent algebra. The operation $4(x \diamond y)$ is well-defined and commutative on B^{++}. Moreover, it is associative if and only if $[[x, y], z] = 0$ holds for all $x, y, z \in B$.

- $[[x, y], z] = 0$ holds if N is 2-step nilpotent.
- This gives non-commutative split C^*-by-nilpotent algebra with associative $4(x \diamond y)$.
- In a C*-algebra, $[[x, y], z] = 0$ implies $[x, y] = 0$, hence commutativity.
- Molnár's 2nd Theorem follows.

Associativity of $4(x \diamond y)$

Theorem (La Rosa, Mancini, GN 2024)

Let $B = \mathcal{A} \oplus N$ be a split C^*-by-nilpotent algebra. The operation $4(x \diamond y)$ is well-defined and commutative on B^{++}. Moreover, it is associative if and only if $[[x, y], z] = 0$ holds for all $x, y, z \in B$.

- $[[x, y], z] = 0$ holds if N is 2-step nilpotent.
- This gives non-commutative split C^*-by-nilpotent algebra with associative $4(x \diamond y)$.
- In a C*-algebra, $[[x, y], z] = 0$ implies $[x, y] = 0$, hence commutativity.
- Molnár's 2nd Theorem follows.

Associativity of $4(x \diamond y)$

Theorem (La Rosa, Mancini, GN 2024)

Let $B = \mathcal{A} \oplus N$ be a split C^*-by-nilpotent algebra. The operation $4(x \diamond y)$ is well-defined and commutative on B^{++}. Moreover, it is associative if and only if $[[x, y], z] = 0$ holds for all $x, y, z \in B$.

- $[[x, y], z] = 0$ holds if N is 2-step nilpotent.
- This gives non-commutative split C^*-by-nilpotent algebra with associative $4(x \diamond y)$.
- In a C*-algebra, $[[x, y], z] = 0$ implies $[x, y] = 0$, hence commutativity.
- Molnár's 2nd Theorem follows.

Associativity of $4(x \diamond y)$

Theorem (La Rosa, Mancini, GN 2024)

Let $B = \mathcal{A} \oplus N$ be a split C^*-by-nilpotent algebra. The operation $4(x \diamond y)$ is well-defined and commutative on B^{++}. Moreover, it is associative if and only if $[[x, y], z] = 0$ holds for all $x, y, z \in B$.

- $[[x, y], z] = 0$ holds if N is 2-step nilpotent.
- This gives non-commutative split C^*-by-nilpotent algebra with associative $4(x \diamond y)$.
- In a C*-algebra, $[[x, y], z] = 0$ implies $[x, y] = 0$, hence commutativity.
- Molnár's 2nd Theorem follows.

Associativity of $4(x \diamond y)$

Theorem (La Rosa, Mancini, GN 2024)

Let $B = \mathcal{A} \oplus N$ be a split C^*-by-nilpotent algebra. The operation $4(x \diamond y)$ is well-defined and commutative on B^{++}. Moreover, it is associative if and only if $[[x, y], z] = 0$ holds for all $x, y, z \in B$.

- $[[x, y], z] = 0$ holds if N is 2-step nilpotent.
- This gives non-commutative split C^*-by-nilpotent algebra with associative $4(x \diamond y)$.
- In a C*-algebra, $[[x, y], z] = 0$ implies $[x, y] = 0$, hence commutativity.
- Molnár's 2nd Theorem follows.

Outline

Lie triple systems

- Lie triple systems are ternary operations $[x, y, z]$ motivated by the double Lie bracket
$$[[x, y], z].$$
- In a Lie group, twisted subgroups correspond to Lie triple subsystems of the tangent Lie algebra.
- If $(L, \bullet)$ is a real analytic Bruck loop, then its tangent space is an (abstract) Lie triple system.
- Baker-Campbell-Hausdorff formula (GN 2002): Near the identity, the Bruck loop operation can be expressed as an infinite series of L.t.s. terms:
$$\log(\exp(x) \bullet \exp(y)) = x + y + \frac{1}{3}[y, x, x] - \frac{1}{6}[x, y, y] + \cdots$$
- This implies BCH formula for the geometric mean.

Lie triple systems

- Lie triple systems are ternary operations $[x, y, z]$ motivated by the double Lie bracket
$$[[x, y], z].$$
- In a Lie group, twisted subgroups correspond to Lie triple subsystems of the tangent Lie algebra.
- If $(L, \bullet)$ is a real analytic Bruck loop, then its tangent space is an (abstract) Lie triple system.
- Baker-Campbell-Hausdorff formula (GN 2002): Near the identity, the Bruck loop operation can be expressed as an infinite series of L.t.s. terms:
$$\log(\exp(x) \bullet \exp(y)) = x + y + \frac{1}{3}[y, x, x] - \frac{1}{6}[x, y, y] + \cdots$$
- This implies BCH formula for the geometric mean.

Lie triple systems

- Lie triple systems are ternary operations $[x, y, z]$ motivated by the double Lie bracket
$$[[x, y], z].$$
- In a Lie group, twisted subgroups correspond to Lie triple subsystems of the tangent Lie algebra.
- If $(L, \bullet)$ is a real analytic Bruck loop, then its tangent space is an (abstract) Lie triple system.
- Baker-Campbell-Hausdorff formula (GN 2002): Near the identity, the Bruck loop operation can be expressed as an infinite series of L.t.s. terms:
$$\log(\exp(x) \bullet \exp(y)) = x + y + \frac{1}{3}[y, x, x] - \frac{1}{6}[x, y, y] + \cdots$$
- This implies BCH formula for the geometric mean.

Lie triple systems

- Lie triple systems are ternary operations $[x, y, z]$ motivated by the double Lie bracket
$$[[x, y], z].$$
- In a Lie group, twisted subgroups correspond to Lie triple subsystems of the tangent Lie algebra.
- If $(L, \bullet)$ is a real analytic Bruck loop, then its tangent space is an (abstract) Lie triple system.
- Baker-Campbell-Hausdorff formula (GN 2002): Near the identity, the Bruck loop operation can be expressed as an infinite series of L.t.s. terms:
$$\log(\exp(x) \bullet \exp(y)) = x + y + \frac{1}{3}[y, x, x] - \frac{1}{6}[x, y, y] + \cdots$$
- This implies BCH formula for the geometric mean.

Lie triple systems

- Lie triple systems are ternary operations $[x, y, z]$ motivated by the double Lie bracket
$$[[x, y], z].$$
- In a Lie group, twisted subgroups correspond to Lie triple subsystems of the tangent Lie algebra.
- If $(L, \bullet)$ is a real analytic Bruck loop, then its tangent space is an (abstract) Lie triple system.
- Baker-Campbell-Hausdorff formula (GN 2002): Near the identity, the Bruck loop operation can be expressed as an infinite series of L.t.s. terms:
$$\log(\exp(x) \bullet \exp(y)) = x + y + \frac{1}{3}[y, x, x] - \frac{1}{6}[x, y, y] + \cdots$$
- This implies BCH formula for the geometric mean.

BCH formula for the Wasserstein mean

Problem

- Is there a BCH formula for the Wasserstein mean in terms of the Jordan product?
- What is $\mathrm{Aut}(B^{+}+)$?

- Recently, Choi, Kim et al. proved binomial expansion formulae for the geometric mean and the Wasserstein mean using the Taylor expansion for the analytic power map.

BCH formula for the Wasserstein mean

Problem

- Is there a BCH formula for the Wasserstein mean in terms of the Jordan product?
- What is Aut$(B^{+}+)$?

- Recently, Choi, Kim et al. proved binomial expansion formulae for the geometric mean and the Wasserstein mean using the Taylor expansion for the analytic power map.

BCH formula for the Wasserstein mean

Problem

- Is there a BCH formula for the Wasserstein mean in terms of the Jordan product?
- What is $\mathrm{Aut}(B^+ +)$?

- Recently, Choi, Kim et al. proved binomial expansion formulae for the geometric mean and the Wasserstein mean using the Taylor expansion for the analytic power map.

BCH formula for the Wasserstein mean

Problem

- Is there a BCH formula for the Wasserstein mean in terms of the Jordan product?
- What is $\mathrm{Aut}(B^{+}+)$?

- Recently, Choi, Kim et al. proved binomial expansion formulae for the geometric mean and the Wasserstein mean using the Taylor expansion for the analytic power map.

THANK YOU FOR YOUR ATTENTION!